A PROCEDURE FOR ELECTING
THE EXECUTIVE COMMITTEE
OF THE GOVERNMENT DEPARTMENT

I assume we aim to elect an executive committee of eight representatives, including at least four full professors, from a single, department-wide constituency by proportional representation. Two types of proportional-representation system are widely used: party-list PR and the Hare System (a.k.a. Single Transferable Vote). The system described here is a version of the latter based on the so-called Senatorial Rules, originally devised for Irish Senate elections (Irish Senates [Proportional Representation] Order, 1921 [Statutory Rules and Orders, 1921, No. 727]). Unlike the simpler Ashtabula version, once widely used in U.S. cities (and still used for University Council elections at U.T.), this one is designed to eliminate (insofar as practicable) all elements of chance, and it is generally favored for small electorates.

I do not contend that the proposed system is immune from criticism or incapable of improvement, only that the Hare System is the one pure P.R. system in widespread use that does not depend on party labels, that the Senatorial Rules constitute the most refined, chance-free version of the Hare System implemented by a real polity, and that it is easier at this point in our department's history to adopt an extant system than to try to reach consensus on some innovation.

I briefly sketch the procedure and its rationale in section 1, fill in some details (without mathematical precision) in section 2, and discuss balloting in section 3. In section 4 I offer a precise mathematical formulation (the first ever, and a prerequisite for computerized ballot counting), and in section 5 I add a proposal for filling vacancies.

1. Sketch of the System

When used to elect eight representatives at-large from a single constituency, the Hare System works as follows: Every voter ranks the candidates in order of preference, specifying his first choice, second choice, third choice, etc. A voter need not rank all candidates: he is free to specify just his first choice, just his first and second choices, or the like. Any candidate receiving more than one-ninth of all first-choice votes is elected. His "surplus" votes--the number he receives beyond the minimum needed for election--are then transferred to other candidates according to his supporters' second preferences. Such transfers might increase the vote totals of additional candidates enough to secure their election. If, after all possible transfers and elections, some of the eight seats remain to be filled, the candidate with fewest first-choice votes is deleted from every ballot, causing his supporters' second choices to rise to first place. This process is reiterated until eight candidates have been elected.

To ensure that at least four representatives are full professors, all candidates who are not full professors are deleted from every ballot once four such candidates have been elected, and when the number of full professors remaining on any ballot is equal to 4 minus the number of full professors so far elected, all full professors who remain on any ballot are declared elected (and then deleted from every ballot before anyone is eliminated).

The rationale for requiring an elected representative to secure more than one-ninth of the total vote is that no more than eight candidates can secure such support whereas it is possible for nine candidates each to secure just one-ninth of the total vote.

Suppose there are thirty-five voters. Then any faction of four voters who prefer a given set of candidates to all other candidates can guarantee the election of some candidate in that set. It is not necessary that all four voters rank the same candidate first.

Surplus votes are transferred to ensure proportional voice. If, for example, eight voters rank Prof. X first but just four of those first-place votes were needed to secure his election, then Prof. X's four "surplus" votes are transferred to other candidates according to the second preferences of his eight supporters. That way a group of eight voters can, in effect, elect two representatives, the number to which they are entitled. Without the transfer, the eight voters could still elect two representatives by coordinating their votes. The transfer procedure eliminates the need for such coordination.

If Prof. X is elected with a surplus and his supporters all rank the same candidate second, that candidate will, of course, receive the entire surplus. But what if Prof. X's supporters differ in their second choices? Which candidates get how much of the surplus? Although different versions of the Hare System handle this case in different ways, they are all based on the same ideal: that Prof. X's surplus should be allocated in proportion to his supporters' second choices--which means, for example, that if one-third of his supporters rank Prof. Y second then Prof. Y is entitled to one-third of Prof. X's surplus. The Senatorial Rules achieve this ideal with mathematical precision. The Ashtabula version approximates the ideal for large electorates, but not for small ones.

2. Details

This sketch omitted a number of details, some of which distinguish the proposed system, based on the Senatorial Rules, from other versions of the Hare System. Here I discuss those details informally (or semi-formally), leaving a full and precise formulation for section 4.

Votes are counted in steps. At each step, we are given a set of ballots--a ballot set, let us call it. Each ballot in the set contains a ranking of candidates and a value (or weight). Candidate x's vote total in a given ballot set is the sum of values of the x-ballots in that set--the ballots on which x is ranked first. At each step, we do three things: decide who, if anyone, is elected at that step; decide whether that step is terminal; and if it is not terminal, transform the ballot set used at that step into the ballot set to be used at the next step (by deleting certain candidates and, in some cases, changing the values of certain ballots).

Some versions of the Hare System assign the value 1 to every initial ballot. The Senatorial Rules (followed here) assign the value 100 to every initial ballot: in effect, every voter casts 100 votes. Say there are n initial ballots. Then the overall vote is n times the value of each initial ballot, and the vote total needed for election, called the Droop Quota, is the smallest integer greater than one-ninth of the overall vote. So if the value of each initial ballot is 1, the Droop Quota is the smallest integer greater than n/9, and if (as in the proposed system) the value of each initial ballot is 100, the Droop Quota is the smallest integer greater than 100n/9. Suppose n = 30. Then if each initial ballot has the value 1, the Droop Quota is 4, but if each has the value 100, the Droop Quota is 334. Because, as you will see, the transfer of surplus votes can reduce ballot values, the Senatorial Rules make it a bit easier for a candidate to achieve the Droop Quota: 334 out of 100n is slightly less than 4 out of n.

Let q be the Droop Quota, and suppose candidate x is elected at a given step with a vote total t>q. Then x's surplus vote, which must be transferred to other candidates, is t-q. Some versions of the Hare System require that t-q of the x-ballots be selected for transfer (to the second-choice candidates marked thereon) at random. Only for large electorates, however, are random samples likely to be fair, or proportional. For small electorates like ours, random transfers would often be grossly disproportional to voter preferences while making electoral outcomes highly sensitive to chance, thereby decreasing the likelihood that a recount of ballots would yield the same result as the first count. The Senatorial Rules require instead that t-q of x's vote total be transferred to other candidates in exact proportion to the number of second-choice votes each candidate has received from x's supporters. Assuming all x-ballots specify second choices, this means that the values of the x-ballots are multiplied by t-q/t, after which x (having been elected) is deleted from every ballot, causing his supporters' second choices to rise to first place. The total thereby transferred is then

$\frac{t-q}{t} \cdot t = t-q$, precisely x's surplus.

To illustrate, suppose q = 400 and x is ranked first on six ballots, each with the value 100. And suppose y is ranked second on one of those ballots, z on two of them, and w on three. Then x is elected with a vote total of 600, hence a surplus of 200 (600-400), and the value of each x-ballot is reduced from 100 to 33 1/3 (200/600 • 100), after which x is deleted from all ballots, causing y to rise to first place on one of the six ballots of x's supporters, z on two of them, and w on three. As a result, y picks up 33 1/3 votes, z 66 2/3, and w 100, for a total transfer of 200 votes. By contrast, the random procedure <u>might</u> have transferred one ballot to y, one to z, and none to w.

If x is elected with vote total t but <u>no</u> x-ballot specifies a second choice, then, of course, no transfer is made. If <u>some but not all</u> x-ballots specify second choices, the values of the x-ballots are multiplied, not by t-q/t, but by t-q/t', where t' is the total value of those x-ballots that do specify second choices. This ensures that the number of votes actually transferred is t-q (x's surplus) rather than a smaller figure.

Now suppose two or more candidates achieve the Droop Quota at step k. According to the Senatorial Rules (and, indeed, practically all versions of the Hare System), only <u>one</u> of those candidates is elected at step k. All who achieve the quota at that step must <u>eventually</u> by elected (unless that prevents four full professors from being elected), but not at step k. <u>Which</u> one is elected at step k? The one with the maximum vote total, if there is just one. If, however, several candidates are <u>tied</u> for maximum vote total, the tie is broken by reducing the set of tied candidates to those among them who had the greatest vote total at step 1, then (if a tie remains) to those with the greatest vote total at step 2, and so on through step k-1. Any remaining tie is broken randomly.

Let me anticipate section 4 by stating this part of the procedure a bit more formally. Where α is a set of candidates and $B_1, \ldots, B_k$ are ballot sets, let $MAX(\alpha, B_1, \ldots, B_k)$ be the result of paring down α to the subset of candidates who each have the maximum vote total in B_1, then paring down this subset to the subset with the maximum vote total in B_2, and so on through B_k. If $B_1, \ldots, B_k$ are the ballot sets corresponding to steps 1 through k, and if at least one member of the set C of candidates listed in B_k has a vote total (at step k) no less than q, then the candidate elected at set k is a randomly chosen member of $MAX(C, B_k, B_1, \ldots, B_{k-1})$ (or the sole member, in case there is just one).

If, at step k, there remain seats to be filled but no candidate has a vote total of at least q, and if there are more candidates listed on ballots at step k than unfilled seats, then (unless this violates the four-full-professor constraint) the candidate with the <u>lowest vote total</u> is <u>eliminated</u> from everyone's ballot, causing his supporters' second choices to rise to first place. But what if two or more candidates are <u>tied</u> for lowest vote total? According to the proposed procedure, based on the Senatorial Rules, such a tie is broken as follows: We first select those tied candidates who had the lowest vote total at step 1, then (if a tie remains) those who had the lowest vote total at step 2, and so on through step k-1. If this does not completely break the tie, we examine the ballots (at step k) on which the surviving tied candidates are ranked first and select those tied candidates with the lowest <u>second</u>-preference vote total in this subset of ballots, then those among the latter candidates with the lowest third-preference total, etc. (Here, and here alone, I deviate from the Senatorial Rules, which do not look beyond second-preference totals, because I see no reason to introduce chance unless necessary.) Only then do we break any remaining tie randomly. Note that this procedure has the effect of first eliminating those candidates, if any, who are ranked first on <u>no</u> ballots.

Let me redescribe this elimination procedure more formally. If α is a set of candidates and $B_1,\ldots,B_k$ are ballot sets, let $MIN(\alpha, B_1,\ldots, B_k)$ be the result of paring down α to the subset of candidates with the lowest vote total in B_1, then paring down this subset to the subset with the lowest vote total in B_2, and so on through B_k, and let $RED(\alpha, B_1)$ be the result of reducing α to the subset of candidates with the lowest (first-preference) vote total in B_1, then reducing this subset to the subset with the lowest second-preference vote total in B_1 (the lowest vote total after all first choices have been deleted from the ballots in B_1), and so on through third-preference totals, etc. If $B_1,\ldots, B_k$ are the ballot sets corresponding to steps 1 through k, a candidate is selected for elimination at step k by narrowing the set C of candidates listed in B_k to $MIN(C, B_k, B_1,\ldots, B_{k-1})$, then to $RED[MIN(C, B_k, B_1,\ldots, B_{k-1}), B_k^*]$, where B^*_k is the set of ballots in B_k whose top-ranked candidates belong to $MIN(C, B_k, B_1,\ldots, B_{k-1})$, and finally making a random choice (if need be) from this last subset.

Because a voter need not rank all available candidates, we might get to a step at which some seats remain to be filled but no candidate can possibly receive a vote total of q. At that point, we continue to eliminate candidates until those remaining on any ballot are equal in number to the unfilled seats, and we declare all of them to be elected.

3. Balloting

Balloting requires constitutional definitions of "eligible voter" and "eligible candidate."

Some eligible voters may be unavailable because they are on leave or choose not to serve. The new by-laws might leave it to an election committee to decide who is available.

I propose that each ballot paper contain an alphabetical list of eligible, available candidates with a blank or box next to each name, plus the following instructions:

> Rank candidates in order of preference by marking "1" next to your first choice, "2" next to your second choice, "3" next to your third choice, etc. You may assign numbers to all candidates, but you need not do so. You may not assign the same number to two or more candidates. If you do, your ranking from that point down will be ignored.

I would collect ballots by giving every voter two envelopes, one marked with the voter's name, the other unmarked, and instructing him to place his ballot in the unmarked envelope and the latter in the marked envelope, sealing both envelopes.

If a voter's ballot assigns the same number k to two or more candidates, all candidates with numbers $\geq$ k should be deleted from that voter's ranking. If all candidates were thereby deleted, the ballot would be invalid. Blank ballots, too, would be invalid. Only valid ballots would be counted in reckoning the Droop Quota.

I would not treat a ballot with nonconsecutive numbering as invalid. If, for example, a voter assigns 1 to Prof. X, 3 to Prof. Y, and 2 to no candidate, I would simply count Y as the voter's second choice.

4. Formal Definition of the System

4.1. Data. The outcome of the proposed system depends on a set of valid ballot papers, a set of candidates, a partition of candidates into full professors and others, and a random choice function, RAND, which chooses one member, RAND(α), from every nonempty set α of candidates. (In case α is a unit set, RAND(α) is, of course, the sole candidate therein.) RAND might be operationalized by drawing lots, using a random-number generator, or whatnot. Details are best left to an election committee.

4.2. Assumption. I assume there are at least eight eligible, available candidates, including at least four full professors, who are each assigned a number on some valid ballot paper.

4.3. Definitions. A precise statement of the vote-counting process requires nine definitions:

> q = the smallest integer > $\frac{100n}{9}$, where n is the number of valid ballot papers.

> A ballot is a paper (or other convenient representation) that

specifies (1) a finite (possibly empty) sequence of candidates, called a _ranking_, and (2) a rational number, called a _value_.

A _ballot set_ is a finite (possibly empty) set of ballots.

An _x-ballot_ is a ballot on which candidate x is ranked first.

The _vote total_ of candidate x in ballot set B in the sum of values of the x-ballots (if any) comprised in B.

C(B) = the set of candidates ranked (at some level) on some ballot in ballot set B.

$MAX(\alpha, B_1, \ldots, B_k)$ is defined, for an arbitrary set α of candidates and arbitrary ballot sets $B_1, \ldots, B_k$, by the following recursion:

$MAX(\alpha, B_1) = \{x \in \alpha \mid$ for every $y \in \alpha$, x's vote total in B_1 is no less than y's$\}$

$MAX(\alpha, B_1, \ldots, B_k, B_{k+1}) = MAX[MAX(\alpha, B_1, \ldots, B_k), B_{k+1}]$

$MIN(\alpha, B_1, \ldots, B_k)$ is defined, for an arbitrary set α of candidates and arbitrary ballot sets $B_1, \ldots, B_k$, by the following recursion:

$MIN(\alpha, B_1) = \{x \in \alpha \mid$ for every $y \in \alpha$, x's vote total in B_1 is no greater than y's$\}$

$$MIN(\alpha, B_1, \ldots, B_k, B_{k+1}) = MIN[MIN(\alpha, B_1, \ldots, B_k), B_{k+1}]$$

RED(α, B) is defined, for an arbitrary set α of candidates and an arbitrary ballot set B, by the following recursion (the recursion variable being the maximum number of candidates ranked at any level on any ballot in B):

If at most one candidate occurs on any ballot in B, RED(α, B) = MIN(α, B).

Otherwise, RED(α, B) = RED[MIN(α, B), B'], where B' is the ballot set obtained by deleting the top-ranked candidate from each ballot in B.

4.4. The procedure. The vote-counting procedure consists of a finite sequence of steps, each corresponding to a ballot set. Let B_i be the ballot set corresponding to step i. To define the procedure, we must define the initial ballot set, B_1, the set of candidates (if any) elected at each step, the terminal step, and for every nonterminal step i, the ballot set B_{i+1} corresponding to step i+ 1:

B_1 = the set of valid ballot papers, each with the value 100.

x is elected at step i if, and only if, $x \in C(B_i)$ and at least one of the following three conditions holds:

(1) x's vote total in B_i is at least q, and

$x = RAND[MAX(C(B_i), B_i, B_1, \ldots, B_{i-1})]$

(2) The number of candidates in $C(B_i)$ is equal to 8 minus the number of candidates elected at steps 1 through i-1.

(3) Every candidate has a vote total less than q in B_i, the the number of full professors in $C(B_i)$ is equal to 4 minus the number of full professors elected at steps 1 through i-1, and x is a full professor.

Step i is <u>terminal</u> if, and only if, someone is elected at step i and the number of candidates elected at steps 1 through i is equal to 8.

If Step i is nonterminal then B_{i+1} is obtained by successively performing the following four operations, and only them, on B_i:

(1) If x is elected at step i with vote total $t > q$, if t' is the total value of those x-ballots in B_i that specify a second-ranked candidate, and if $t' > 0$, multiply the value of every x-ballot in B_i by $\frac{t - q}{t'}$.

(2) Delete, from every ballot in B_i, any candidate elected at step i.

(3) If four non-full-professor candidates have been elected at steps 1 through i, delete every non-full-professor candidate from every ballot in B_i.

(4) If no one is elected at step i, delete, from every ballot in B_i, RAND(RED[MIN(C(B_i), B_i, B_1, ..., B_{i-1}), B_i^*]), where $B_i^* = \{E \in B_i \mid E$ is an x-ballot for some $x \in$ MIN(C(B_i), B_i, B_1, ..., B_{i-1})$\}$.

A candidate is <u>elected</u> if elected at some step.

Given our assumptions, the procedure must eventually terminate with the election of exactly eight candidates, including at least four full professors.

5. Vacancies

What to do if a vacancy occurs?

To choose a replacement by applying some majoritarian rule to the original set of ballots (or a new set) would violate the objective of proportional representation. Nor is there any obvious, straightforward way to choose a ninth "runner-up" candidate when election results are computed. Once eight candidates have each received a vote total no less than q, it is impossible for a ninth candidate to receive a vote total of q at any subsequent step. Besides, we may not know in advance whether a replacement must be a full professor.

My proposal: When a vacancy occurs, delete the representative who vacated his seat (along with anyone else no longer available) from every valid initial ballot, then re-run the vote-counting procedure until a candidate (or a full-professor candidate, in case only three full professors remain on the committee) is elected who was not originally elected, and fill the vacancy with him. If two or more such candidates are elected at the same step i, fill the vacancy with RAND[MAX(α B_i, $B_1, \ldots, B_{i-1}$)], where $\alpha = C(B_i)$ if four full professors remain on the committee, and otherwise $\alpha = \{x \in C(B_i) \mid x$ is a full professor$\}$.

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