Consequences of Completeness for First Order Logic

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September 23, 2003

1 Basic Tools: Compactness and Lowenheim Skolem

The principal model theoretic tool and consequence of the completeness theorem is the compactness theorm.

- Compactness Theorem:
  If every finite subset of Γ has a model (is satisfiable) then Γ has a model.

The proof follows very simply. Let Γ be any set of sentences. Suppose every finite subset Γ' of Γ is satisfiable. This is equivalent to saying:

$$\Gamma' \not\models \phi \wedge \neg\phi$$

And by the contraposition of the Soundness Theorem we get:

$$\Gamma' \not\vdash \phi \wedge \neg\phi$$

which just says that each Γ' is consistent. Now suppose the Γ as a whole is not satisfiable. This means that Γ does not have a model. By Completeness (in the form that any consistent set of formulas has a model) Γ is inconsistent. But then we must have a deduction of a contradiction from Γ and since deductions are finite, from a finite set of premises in Γ. This entails that some finite subset of Γ is inconsistent, which we showed earlier was impossible.

- Downward Lowenheim Skolem Theorem:
  If T is a theory in a language L such that the set of nonlogical constants in L has cardinality κ, then if T has a model, T has a model of cardinality $\leq \kappa + \omega$.

Proof: Since T has a model, T is consistent. So we can apply the technique of the Completeness theorem to T and construct a Henkin model $\mathcal{A}^T$ for T. The objects in $\mathcal{A}^T$ consist of equivalence classes of L terms, of which there are perforce $\leq \kappa + \omega$.

- Upward Lowenheim Skolem Theorem:
  If T is a theory that has a model with cardinality $\omega_\alpha, \alpha \geq 0$, then T has a model of cardinality cardinality $\kappa > \omega_\alpha$.

Proof: Extend the language of T by introducing a set of distinct constants $c_1, c_2, \ldots c_\kappa$ where this set has cardinality κ. Now consider the set

$$\Sigma = T \cup \{c_i \neq c_j : i \neg j, i, j < \kappa\}$$

It is clear that every finite subset of Σ has a model, and hence by Compactness, Σ itself has a model. But this model must satisfy all the inequalities in Σ, which ensures that it has cardinality κ.

2 Consequences of Compactness and Lowenheim Skolem

Actually the Upwards Lowenheim Skolem theorem is a consequence of Compactness. Additional consequences are the following:

- The language of ZFC + CH (axiomatic set theory with choice and the continuum hypothesis) contains just one nonlogical constant '$\in$'. So by Downward Lowenheim Skolem, there is a countable model of ZFC + CH. This clashes with "intuitions" that the model set theory is describing is much "bigger" (at least as big as the first strongly inaccessible cardinal which is much bigger than ω_n, let alone ω_0.
- The set theoretic reconstruction of the reals using Dedekind cuts or Cauchy sequences is formalizable in set theory together with nonlogical

symbols for 0, successor, + and $\times$. So the theory describing the reals also has a countable model, whereas intuitions and Cantor's theorem strongly suggest that the cardinality of the reals is $> \omega_0$.

- By the Upward Lowenheim Skolem Theorem, the theory of arithmetic has uncountable models.

These results suggest that there are difficulties for a certain version of the philosophical position of logicism, where the latter is understood as maintaining that our intuitions about mathematical structures can all be captured via theories formulated in first order logic. In particular, we cannot do justice to our intuitions about cardinality. Further, we can use compactness to show that many theories do not determine the intended mathematical structure up to isomorphism.

Consider for instance the theory of first order Peano Arithmetic (which contains axioms for successor and a scheme for induction). Call this theory PA. Now consider the following set

$$\Sigma = T \cup \{x > n : n \in \omega\}$$

which we can write within PA as follows:

$$\Sigma = T \cup \{x > t : t = 0 \vee \exists y t = s(y)\}$$

Every finite subset of Σ is evidently satisfiable. So by Compactness, Σ is satisfiable. But the Henkin model of Σ, $\mathcal{A}^\Sigma$ contains not only the natural numbers but a dense set of copies of the integers on "top" of the "standard" numbers. These copies of the integers are generated by predecessors and successors of multiples of some object c that is bigger than every standard number. Note that the size of this model is still countable, as we can construct a bijection from A^Σ to ω as follows: $f(n) = 2^n$, for each finite successor of 0 in A^Σ, whereas f maps all finite predecessors of c to powers of 3, all successors of c to powers of 5, all predecessors of $2c$ to powers of 7 and so on. Since there are an infinite number of primes and of distinct powers of each prime, this mapping is a bijection. But notice that $\mathcal{A}^\Sigma$ is not isomorphic to the standard model of the naturals. For any y the set $\{x < y\}$ is finite in N but this is not so in $\mathcal{A}^\Sigma$. So no bijection f can satisfy the constraint:

$$\mathcal{N} \models x < y[a, b] \text{ iff } \mathcal{A}^\Sigma \models x < y[f(a), f(b)]$$

Other applications of compactness show that “finitude” is not definable by a first order formula, indeed by any set of first order formulas.