

The *Tractatus* on Inference and Entailment

Ian Proops

The University of Michigan

*Introduction*¹

The aim of this essay is to investigate Wittgenstein's well-known, yet obscure, objection to the views of Frege and Russell on deductive inference and its justification.

The objection, such as it is, is set out at *Tractatus*² 5.132:

If p follows from q , I can make an inference from q to p ; deduce p from q .
The method of inference is to be gathered from the two propositions alone.
Only they themselves can justify the inference.
“Laws of inference”³ [*Schlussgesetze*], which—as in Frege and Russell—are to justify the inferences, are senseless [*sinnlos*] and would be superfluous.
(My translation.)

The passage raises two immediate questions. First, what does Wittgenstein mean by “laws of inference”? And, secondly, why would Frege and Russell have thought of these laws as involved in the justification of inference? I will take these questions one at a time.

[1] *Laws of inference*

The majority view among scholars who have addressed the first question is that when speaking of “laws of inference” Wittgenstein means to be referring to the inference *rules* of a formal system.⁴ But, whatever the merits of this suggestion, it faces the immediate difficulty that neither Frege nor Russell understood the term “laws of inference” so

narrowly as to refer *only* to inference rules. In considering whether the basic laws of logic ought to be called “axioms,” Frege says:⁵

Traditionally, what is called an axiom is a thought whose truth is certain without, however, being provable by a chain of logical inferences. The *laws of logic*, too, are of this nature. Some people may nevertheless be inclined to refrain from ascribing the name ‘axiom’ to these *general laws of all inference*, [*allgemeinen Gesetze alles Schließens*] but rather wish to reserve it for the basic laws of a more restricted field. (my translation and emphasis)

Here, clearly, Frege is treating the expression “general laws of all inference” as a variant upon “laws of logic”.

That Russell took “laws of inference” to encompass more than inference rules is suggested by a passage in “Necessity and Possibility,” a paper he gave to the Oxford Philosophical Society in 1905. He writes:⁶

There are certain general propositions, which we may enumerate as the *laws of deduction*: such are “if not-p is false, then p is true”, “if p implies not-q, then q implies not-p” [and] “if p implies q and q implies r, then p implies r”; in all we need about ten such principles.... (Emphasis my own.)

Because Russell says in a footnote to § 13 of the *Principles* that “[he does] not distinguish between inference and deduction,” it seems reasonable to suppose that the expression “laws of *deduction*” figures in this passage merely as a stylistic variant upon “laws of *inference*.” But if so, then Russell, too, would appear to be taking “laws of inference” to include logical laws.

(Notice that in both of these passages when Frege and Russell speak of “the laws of logic” or “the laws of deduction” they have in mind specifically *primitive* or *basic* logical laws. Since at other times they use these expressions to refer to the primitive laws *and* theorems of their systems, we shall need to remain alert to this ambiguity.)

[2] *The justification of inference*

Assuming that Wittgenstein is using “laws of inference” to refer to what Frege and Russell mean by this term, our second question becomes: How exactly, in Wittgenstein’s view, do Russell and Frege see logical laws (and possibly inference rules⁷) as involved in the justification of inference? and what does Wittgenstein find objectionable in this idea? I shall offer my own answers to these questions shortly, but first I want to examine a rival account that has been offered by Thomas Ricketts in his paper “Frege, the *Tractatus* and the Logocentric Predicament.”⁸ Ricketts takes Wittgenstein’s criticism to be meant to apply in one way to Russell and in quite another to Frege. Each suggestion has considerable initial appeal, but I will argue that *as exegesis* neither is ultimately successful.

Against Russell, the criticism is supposed focus on (what Ricketts alleges to be) Russell’s failure adequately to distinguish inference rules from logical laws.⁹ As Ricketts sees the matter, this confusion is bound up with a mistaken view of the way in which (what *we* would recognize as) inference rules serve to license the inferences they govern. According to Ricketts, Russell sees inference rules—which for Ricketts’s Russell are not to be distinguished from the laws of logic—as conferring their licence only by serving as *premises* in arguments.¹⁰ Accordingly, Ricketts presents Wittgenstein as seeking to expose the mistake in such a view by means of a (tacit) appeal to Lewis Carroll’s well-known regress argument.¹¹

To illustrate: suppose *modus ponens* were taken to function as a premise. Then the argument from p and ‘if p then q ’, to the conclusion q would actually take the following form:

$$\begin{array}{l} p \\ p \supset q \\ (p \cdot p \supset q) \supset q \\ \text{Therefore, } q \end{array}$$

Here *modus ponens* is supposed to have been formulated as a proposition rather than a rule. But, in this role it will be powerless to license the inference to the conclusion q , since the argument is now of a new form, one not covered by *modus ponens*. Nor may we remedy the situation by appealing to some new inference rule, because its incorporation as a premise will again alter the overall shape of the argument.

Ricketts concedes that this story cannot be the whole explanation of Wittgenstein’s critique of “laws of inference,” since there is no reason to suppose that *Frege* confused inference rules with axioms. He therefore offers a separate account of how the criticism of 5.132 is supposed to apply to *Frege*. I will turn to this account in a moment, but first I want to raise some doubts about the way the criticism is supposed to apply to *Russell*.

My main concern is that Ricketts’s interpretation portrays Wittgenstein as guilty of an uninterestingly uncharitable reading of *Russell*. For *Russell* was in fact perfectly clear about the distinction between inference rules and logical laws. Indeed, as we shall see, he was clear about this distinction precisely because he saw that a failure to respect it would lead directly to *Carroll*’s regress.

In the course of an early proof in *Principia Mathematica* Russell and Whitehead remark: “The principle of deduction gives the general rule according to which the inference is made, but is not itself a premise in the inference. If we treated it as a premise, we should need either it or some other general rule to enable us to infer the desired conclusion, and thus we should gradually acquire an increasing accumulation of premises without ever being able to make any inference.”¹² And in the next paragraph, having mentioned that principles of deduction may also function as premises in an inference—that is, when they no longer play their inference-licensing role—they continue: “This distinction between the two uses of principles of deduction is of some philosophical importance, and in the above proofs we have indicated it by putting the rule of inference in square brackets.”¹³ So for Russell and Whitehead the distinction between the use of a principle of deduction as a premise and its use as a rule is important enough to be embodied in the very symbolism of *Principia*.¹⁴

Making the same point in the earlier *Principles of Mathematics*, Russell says: “We may observe...that in a particular inference, the rule according to which the inference proceeds is not required as a premise.”¹⁵ And he goes on to argue that if we were to require the rule to function as a premise—if we were to do what is “formally necessary”¹⁶—we would be faced with a Carroll-style regress. Russell concludes that the rule according to which the inference proceeds should be seen as “a respect in which the formalism breaks down,” and he recommends that such rules be accorded the status of informal principles that are employed in making inferences but not recorded as lines in proofs.¹⁷

Ricketts does not say why he takes Russell to have confused inference rules with logical laws, but I suspect that his reading is based on at least one of the following

considerations. First, in section 18 of the *Principles*, Russell applies the term “axiom” to inference rules and logical laws indiscriminately.¹⁸ This might seem to suggest that he is insensitive to the differences between them. It ought not to mislead us, however, for even while speaking this way, Russell is at pains to emphasize the peculiar status of inference rules among axioms. Thus in section 18, while he lists several axioms in the form of statements containing implicitly quantified variables (for example, (6): “If p implies q and q implies r , then p implies r ”), he formulates the one “axiom” that we would call an inference rule very differently:

(4) A true hypothesis in an implication may be dropped, and the consequent asserted. This is a principle incapable of formal symbolic statement, and illustrating the essential limitations of symbolism.

The rule is recorded as a *permission*, rather than as a statement of fact, and its anomalous status is clearly flagged. There may be room for debate about what it means to say that the principle is “incapable of formal symbolic statement,” but a comparison with *Principles* § 45 strongly suggests that it means that such rules cannot be recorded as premises in the deductions in which they are used. It seems likely, therefore, that by calling *modus ponens* an indemonstrable “axiom” Russell merely wishes to emphasize that it forms part of the fundamental basis of mathematics, not that it is to be construed as functioning as a *premise* in arguments.

What I think is most likely to be driving Ricketts’s interpretation, however, is Russell’s speaking in *Principia* of one and the same “principle of deduction” as having two kinds of *use*: a “use for implication” and a “use for inference.”¹⁹ But, so long as one is clear—as Russell was—that a principle can be “used for inference” without functioning

as a premise in that very inference, this way of conceiving the matter will not lead to a Carroll—style regress.

Now, since the relevant passages occur quite early in the *Principles* and in *Principia*, and because we know Wittgenstein to have read both works, it seems plausible that he would have known of Russell’s awareness of the need for a distinction between inference rules and logical laws. Certainly, he *must* have seen the passage quoted from Principles section 18, since the criticism of 5.5351, which focuses on the *Principles*’ (sporadic) practice of prefixing “ $p \supset p$ ” as an antecedent to logical laws, relates directly to this section. So it seems doubtful that this is the criticism Wittgenstein has in mind. Of course, it is always possible that Wittgenstein might have simply misread, or skimmed over, the relevant passages—or that having read them, he should have forgotten about them—but if that were so, his criticism would lose much of its interest. If we want to find an interpretation of 5.132 that presents Wittgenstein’s objections as having real bite against a position that some philosopher actually held, we will need to seek its target elsewhere.

This brings us to Ricketts’ reconstruction of Wittgenstein’s criticism of Frege. The reading begins from the observation that for Frege reasoning from certain non-logical propositions to others may need to be²⁰ mediated by laws of logic, if we are to make fully explicit the grounds on which the conclusion rests. Consider, for example, the following argument:²¹

- i) All whales are mammals;
- ii) All mammals are vertebrates;

Therefore, iii) All whales are vertebrates.

According to Ricketts's Frege, i) and ii) do not exhaust the grounds for the assertion of iii). Rather, iii) rests additionally upon certain laws of logic. So in order to render the grounds of our inference fully explicit, we shall first need to prove the conclusion iii), by deriving the following logical law from basic laws of logic:

iv) $(\forall F)(\forall G)(\forall H)[(\forall x)(Fx \supset Gx) \supset ((\forall x)(Gx \supset Hx) \supset (\forall x)(Fx \supset Hx))]$.

We then proceed to instantiate the second-order variables in iv), to obtain:

v) $(\forall x)(x \text{ is a whale} \supset x \text{ is a mammal}) \supset ((\forall x)(x \text{ is a mammal} \supset x \text{ is a vertebrate}) \supset (\forall x)(x \text{ is a whale} \supset x \text{ is a vertebrate}))$.

The conclusion, iii), then follows from i), ii) and v) by two applications of *modus ponens*.

Ricketts takes 5.132 to be opposing the idea that logical laws are ever needed to mediate inferences in this way. His view certainly provides a plausible sense in which logical laws might be thought to “justify” inferences; for on this account they do so because they form part of the basis for the assertion of the conclusion. It is far from clear, however, that this could be the notion of “justification” Wittgenstein is focusing on at 5.132.

To see this point it is important to appreciate that on Ricketts's reading the dispute between Wittgenstein and Frege is, at bottom, a disagreement over the true *grounds* for the assertion of the conclusion. For Frege, but not for Wittgenstein, our knowledge of logical laws plays a role in mediating inferences between statements within a particular special science. This means that logical laws will have to be cited in giving a full inventory of the grounds on which a particular inference rests. So if Ricketts is right, we should expect Wittgenstein, in correcting Frege, to deny that laws of logic are among the implicit grounds of the inference, and to insist that the true grounds for the judgement of (iii) are just what they seem to be, namely: i) and ii). That is, we should expect him to say that it is the truth of these propositions, and of these alone, that provides the justification for the inference.

What Wittgenstein actually says, however, is that laws of inference are superfluous because the justification is afforded by premise *and conclusion* alone. So whatever *Wittgenstein* means by the "justification of an inference," it must be something in which the *conclusion* itself can figure. But since the conclusion will not figure among the grounds of any inference worth making, the dispute about "justification" cannot, after all, be one about what ought to be included in a full articulation of an inference's grounds. So although Ricketts certainly identifies *a* sense in which logical laws might be held to be involved in the justification of inferences, it cannot be the sense that is relevant to the interpretation of 5.132.

[3] *Wittgenstein's true target*

To locate the true target of Wittgenstein's criticisms I suggest we begin by considering his views on what *does* justify inference, for this will give us a sharper sense of the notion of "justification" that is in question at 5.132. Before we address this issue, however, we should consider Wittgenstein's use of the term "inference." As is widely known, Frege and Russell hold that it is impossible to draw an inference from a false proposition.²² Wittgenstein disagrees: "One can *draw inferences* from a false proposition." (4.023). He is able to maintain this view because he does not share Frege's conception of inference as the *recognition* of the truth of one proposition on the basis of the recognition of the truth of another. For Wittgenstein, inference is something more akin to the assertion of one proposition's truth on the basis of the *supposition* of the truth of another. Consequently, for Wittgenstein what justifies my inferring q from p, in *his* sense of "inferring," is the fact that q follows from p. And what *this* consists in is nothing more than the obtaining of an internal relation between the forms of p and q. He says:

If the truth of one proposition follows from the truth of others, this expresses itself in relations in which the forms of these propositions stand to one another, and we do not need to put them in these relations first by connecting them with one another in a proposition; for these relations are internal, and exist as soon as, and by the very fact that, the propositions exist. (5.131)

The remark suggests that what Wittgenstein means to be challenging at 5.132 is a view about what it is for two propositions to stand in the relation of logical entailment.²³

Wittgenstein takes logical entailment to consist in an "internal" relation between the "forms" of propositions, while his opponent takes it to consist in something more—something Wittgenstein characterizes as "[the putting of the propositions] in these relations.... by connecting them with one another in a proposition." This remark is darkly metaphorical, and much of this essay will be taken up with illuminating it. The point of

immediate importance, however, is that this view of what is at issue between Wittgenstein and his opponents fits well with the positive suggestion of 5.132, namely, that what justifies a particular deductive inference is *nothing beyond the premise(s) and conclusion themselves*. For since the relation between the forms is an *internal* relation, its holding consists in nothing more than the propositions' *being the very propositions that they are*.²⁴ (I shall return to this point later.)

But if, as I am suggesting, the issue joined at 5.132 concerns the nature of the relation of logical entailment, then why is it presented as a dispute about what is involved in *justifying* an inference? The answer, I believe, is that the notion of "inference" in play at this point in the text is *Wittgenstein's*. This means that the question what justifies the inference from p to q is equivalent to the question what would justify us in asserting that q on the basis of the supposition that p. And *this* question is answered by citing whatever fact one takes the relation of logical entailment to consist in.

If these thoughts are on the right lines, then Wittgenstein would seem to be criticizing Frege and Russell for having given accounts of logical entailment that somehow involve an essential appeal to logical laws (and possibly to inference rules). But do Frege and Russell offer such accounts? I shall argue that in the case of Russell the answer is a clear "yes," and in the case of Frege, a more qualified one.

Russell offers no explicit account of logical entailment in the *Principles*, but when a little later on he is forced to think about how to characterize this relation he develops precisely the kind of account we might have expected, given the conceptual resources available to him in this early work. The account appears in the 1905 paper "Necessity and Possibility," from which I quoted earlier. In this paper Russell is attempting to put flesh

on the bones of the deflationary account of necessity given by G. E. Moore in his 1900 essay: “Necessity.”²⁵ Russell explains Moore’s view thus:

According to this theory a proposition is more or less necessary in proportion as there are more or fewer other propositions to which it is logically prior, p is logically prior to q if q implies p but p does not imply q.²⁶

In other words, a proposition is more necessary in proportion as there are more propositions which imply it but which it does not imply. So, for instance, “Someone is clever” is more necessary than “Russell is clever,” since anything “Someone is clever” implies is already implied by “Russell is clever,” but not conversely.

Russell recognizes that this account will have to employ a notion of implication other than material implication. A false proposition materially implies all propositions and a true proposition is materially implied by every proposition; so using material implication in the account will dictate that all truths are on one level of necessity, and all falsehoods on another.

Because of this problem Russell is prompted to explore the concept of logical entailment, which he terms “deducibility”—and on one occasion²⁷ “logical deducibility.” He sees this as an ordinary concept in need of precise definition. He views his task as one of demarcating a class of material conditionals—“implications,” as he calls them—in which the consequent is intuitively felt to be a necessary consequence of the antecedent. The relevant cases are those in which the truth of the material conditional, “p implies q,” is known independently of the falsity of p and the truth of q. He says:

... in the practice of inference, it is plain that something more than implication must be concerned. The reason that proofs are used at all is that we can sometimes perceive that q follows from p, when we should not otherwise know that q is true; while in other cases; “p implies q” is only to be inferred either from

the falsehood of p or from the truth of qWhat we require is a logical distinction between these two cases. (Ibid., p. 515)

Russell is pointing to the need to distinguish the fact that p materially implies q , from the fact that q is logically entailed by, or in his terminology “deducible from,” p . He defines the latter notion as follows.

We may then say that q is *deducible* from p if it can be shown *by means of the above principles* [i.e. the axioms and inference rules of Russell’s system] that p implies q . (Ibid.)

So q is deducible from p just in case ‘ $p \supset q$ ’ is derivable in Russell’s system.

Russell observes that the definition captures the extension of the concept he is seeking to define:

This meaning of *deducible* is purely logical, and covers, I think, exactly the cases in which, in practice, we can deduce a proposition q from a proposition p without assuming either that p is false or that q is true. (Ibid.)

Finally, having defined the notion of “deducibility,” Russell uses it to explain the intuitive notion of a valid argument. He writes:

It is noteworthy that, in all actual valid deduction, whether or not the material is of a purely logical nature, the relation of premise to conclusion, in virtue of which we make the deduction, is one of those contemplated by the laws of logic²⁸ or deducible from them (Ibid., p. 517).

Notice that Russell regards the relevant material implication as *licensing* the inference.

He calls it the relation “in virtue of which” we make the deduction. As long as the implication is one that holds for reasons other than the falsity of the antecedent or the truth of the consequent, it will licence inferences from antecedent to consequent. Such

cases are those in which the obtaining of the implication is grounded in the laws of inference. In such cases the implication obtains either because it is “contemplated” by the (primitive) laws of logic (that is, is an instance of one of the axioms), or because it can be derived from these laws using the inference rules of Russell system.²⁹

Although the 1905 paper containing Russell’s account of deducibility is a relatively obscure source, the view it contains is suggested by passages in Russell’s more public writings.³⁰ At the beginning of part I of *Principia*, in a section entitled “The Theory of Deduction,” Russell characterizes the subject treated in *Principia* as “the theory of how one proposition can be inferred from another.”³¹ He says that “in order that one proposition may be inferred from another it is necessary that the two should have that relation which makes one the consequence of the other.” So, given this characterization of the subject treated in *Principia*, it seems plausible that Russell takes the theorems of his system to report the obtaining of the *logical entailment* relation. In *Principia* Russell misleadingly calls this relation “implication,” but in view of the clarity shown in his 1905 paper about the distinction between implication and deducibility it seems likely that what he has in mind here is the restriction of the implication relation to those implications that can be shown to hold on the basis of logic alone. Such implications are, after all, just the theorems of the “theory of deduction.” If this is correct, then Russell’s view would seem to be that the relation that justifies inference is just the relation of logically grounded implication. (I shall call an implication ‘logically grounded’ just in case it is derivable as a theorem of Russell’s system.)

These later remarks of Russell suggest that he is no longer thinking of the definition given in the 1905 paper as merely capturing the extension of the logical entailment relation. Rather, he now seems to be regarding the relation of logical

entailment as *consisting in* a special kind of implication. In this connection it is worth noting that after 1905, Russell no longer treats the relation that obtains between premise and conclusion in a sound and valid argument as a primitive relation—the so-called “*therefore*” relation of the *Principles*.³² I would conjecture that this is because Russell has by this point come to regard the 1905 notion of logically grounded implication as providing for a deeper analysis than the 1903 view. (By 1905 the *therefore* relation need no longer be treated as a primitive, since it turns out to be analysable in terms of the notions of logically grounded implication and truth.)

Russell says that the premises of the theory of deduction must contain “as many of the properties of implication as are necessary to *legitimate* the ordinary procedure of deduction.” (*Principia* section A, p. 90, my emphasis). So, assuming, as I have done, that Russell really means to be speaking here of properties of the logical *entailment* relation, we get a clear statement of the view that I have suggested Wittgenstein means to be opposing. For Russell, the laws of inference (understood as the *primitive* logical laws and inference rules of his system) justify inferences because they logically ground the relation of material implication that obtains between premise and conclusion in a valid argument. Strictly speaking, it is the *fact* that this relation is logically grounded that justifies the inference, but because (primitive) laws of inference provide the grounding in question, they can be said to justify inferences in a derivative sense. By contrast, Wittgenstein will say that the “ordinary procedure of deduction” is justified not by the relation of implication between premise and conclusion being one that is counted logically grounded by Russell’s “theory of deduction,” but by an internal relation between the forms of the propositions involved.

Because the envisaged grounding is grounding by the laws of Russell's system, Russell's account of deducibility turns out to be a system-relative one. What it is for p to entail q is for ' $p \supset q$ ' to be derivable from the axioms and inference rules of *Russell's system*. I suspect, however, that Wittgenstein would not have regarded this feature of Russell's view as irremediably problematic. For he appears to have regarded Russell as having a non-system-relative conception of a primitive proposition of logic (or logically primitive inference rule), viz. as one having the highest degree of psychological self-evidence.³³ Reading him in this way, Wittgenstein may have charitably reconstructed Russell's view so that logically grounded implications were understood to be just those that are derivable from intrinsically *primitive* logical laws and inference rules.

A Fregean view that might be the target of Wittgenstein's criticism is suggested in a passage from "Foundations of Geometry: Second Series," where Frege is discussing, not logical entailment exactly, but the related notion of the "dependence" of one thought on a group of others.³⁴ He writes:

Let Ω be a group of true thoughts. Let a thought G follow from one or several of the thoughts of this group by means of a logical inference such that apart from the laws of logic, no proposition not belonging to Ω is used. Let us now form a new group of thoughts by adding the thought G to the group Ω . Call what we have just performed a logical step. Now if through a sequence of such steps, where every step takes the result of the preceding one as its basis, we can reach a group of thoughts that contains the thought A , then we call A dependent upon group Ω . If this is not possible, then we call A independent of Ω .³⁵

Frege offers a precise characterization of the dependence of a thought on a group of true thoughts, Ω , in terms of the yet-to-be-clarified notions of "logical inference" and "law of logic." The notion he defines corresponds closely to the idea of provability in an axiom system with sound inference rules, where the axioms are the thoughts in Ω together with

the laws of logic. Frege offers what is in effect a (proto-) “proof-theoretic”³⁶ account of what it is for one *true* thought to “follow from” another. An account, that is, not of the relation of logical entailment, but of the restriction of that relation to the class of true thoughts. As Frege admits, this is only a *sketch*: further work will need to be done to clarify the notions of “logical inference” and “law of logic.”³⁷

The need for further work could be avoided if we were to *stipulate* certain inferences and laws to be primitively logical, but then we would have defined only a notion of dependence-relative-to-a-stipulation.³⁸ On the other hand, there is reason to believe that Frege might have entertained the hope of working toward a non-relativized, or “absolute,” conception of *dependence*. For, as Tyler Burge and Robin Jeshion have recently (independently) argued, there is good reason to think that Frege believed in the existence of laws of logic and rules of inference that were by their very nature *primitive*.³⁹

Briefly, Burge and Jeshion each claim that, quite apart from any psychological notion of “self-evidence,” Frege employed a non-psychological notion which had to do with a proposition’s place in a hierarchy of truths, structured by objective relations of justification. A truth is “self-evident” in this non-psychological sense if—in Jeshion’s phrase — it is “self-supporting.” That is to say, if by its very nature it derives its justification from no other truths, but contains its own evidence or support within it. When this is so, the truth in question is in an *absolute* sense “unprovable,” but in virtue of its essentially self-grounding character it may be capable of grounding other truths.⁴⁰ According to Jeshion, a thought’s being psychologically obvious is good evidence that it is self-supporting, but self-supportingness is not constituted by psychological obviousness. And, most importantly for our purposes, a truth occupies its position in the objective hierarchy quite independently of our decision to choose it as an axiom.⁴¹

If something like this story is correct—and Burge and Jeshion have made it very plausible that it may be,⁴² then Frege would, after all, have had at his disposal a conception of intrinsically primitive logical laws and principles of inference.⁴³ Thus equipped, he could have aspired to frame a notion of the *objective* dependence of one truth upon another. The idea would be to say that a thought q objectively depends on another p , if q can be derived from p together with primitive logical laws, using only primitively logical rules of inference.

It is debatable how optimistic Frege would have been about the prospects of clarifying the notions of a primitively logical law and a primitively logical inference,⁴⁴ but for our purposes what matters is that *Wittgenstein* might well have taken Frege to have been fully committed to such a project. He certainly seems to have anticipated Burge and Jeshion in taking seriously the idea that Frege was committed to a category of *essentially basic* logical laws. For such a view is the target of 6.127, which runs: “All propositions of logic are of equal status: it is not the case that some of them are essentially basic laws [*wesentlich Grundgesetze*] and others derived propositions” (my translation⁴⁵). Wittgenstein’s reference to “*Grundgesetze*” strongly suggests Frege as the target of this remark; so whether or not Wittgenstein was right to think that Frege took the degree of (psychological) self-evidence—that is, obviousness—as the⁴⁶ criterion of the primitively logical (6.1271),⁴⁷ he does seem to have ascribed to Frege a conception of the basic laws of logic as having a special status—as *essentially basic*.

The important point for our purposes, then, is that *as Wittgenstein viewed him*, Frege would have been able to say that a true thought q is dependent (in an absolute sense) on another p just in case a proof of q from p can be constructed, each line of which is a member of Ω' , where Ω' is the closure under the “primitive logical consequence”

relation of the union of $\{p\}$ with the set of the primitive logical laws. Wittgenstein could then have seen such an account as easily amendable to yield an account of entailment: one simply understands the notion of proof in terms of Wittgenstein's notion of inference instead of Frege's. (That is to say, one does not require that p be true). The resulting account would portray the laws of inference (both logical laws and inference rules) as essential to the justification of inferences. For it would present the question whether an inference is justified as turning upon the possibility of linking together the thoughts expressed by the premise and conclusion by a chain of thoughts constructed in the way described above.

Whether Wittgenstein would have known of these ideas of Frege is, admittedly, unclear. We know that he visited Frege and had philosophical discussions with him in the summer of 1911, toward the end of 1912, and shortly before Christmas 1913; so the issue could have arisen in conversation.⁴⁸ We also know that "Foundations of Geometry: Second Series," was published in the *Jahresbericht der Deutschen Mathematiker-Vereinigung* in 1906, but it is not one of the works that Wittgenstein discusses, or mentions having read.

On the other hand, we do have reason to think that by the time of writing the *Tractatus* Wittgenstein had read Frege's *Grundlagen*, and this work contains at least one passage which gestures toward the kind of account we have seen outlined above. In the course of an argument against the adequacy of inductively established arithmetical laws—an argument whose details need not concern us—Frege observes:⁴⁹

Instead of linking our chain of deductions direct to any matter of fact, we can leave the fact on one side, while adopting its content in the form of a condition. By substituting in this way conditions for facts throughout the whole of a train of

reasoning, we shall finally reduce it to a form in which a certain result is made dependent on a certain series of conditions.

So when we have a derivation of $p \supset q$ from basic logical laws, we have a case where the chain of deductions beginning with p and ending with q has been “reduced to the form where” q is *dependent* on p . But since Frege is here in effect (tacitly) assuming one direction of the deduction theorem: (viz., if $p \mid\text{---} q$, then $\mid\text{---} p \supset q$), this means that for him q is dependent on p whenever q is derivable from p . Moreover, if we regard Frege here as envisaging the process of “substituting conditions for facts” and so on to be reversible, (and so as in effect tacitly committed to both directions of the deduction theorem), we shall have a case of the notion of dependence being cashed in terms of derivability: q is dependent on p iff $\mid\text{---} p \supset q$, that is, iff $p \mid\text{---} q$. So the kind of “proof-theoretic”⁵⁰ account of dependence we have examined above is at least hinted at as early as *Grundlagen*—a work which the Wittgenstein of the *Tractatus* almost certainly read.⁵¹

[4] *Wittgenstein’s criticisms*

I find it plausible that Wittgenstein’s criticisms at 5.132 are directed against something like the view of the logical entailment relation suggested by these writings of Frege and Russell. For, as I have already argued, they are directed against a certain view of the relation that must obtain between two propositions, p and q , if we are legitimately to infer q from p in *Wittgenstein’s* sense of “infer,” and these views of Frege and Russell suggest one account of what this relation might be.

Wittgenstein's main objections to this view are stated at 5.132. First and most fundamentally, that in the hands of Frege and Russell this account of logical entailment rests on a faulty conception of the laws of logic as fact-presenting statements. For Wittgenstein, there are no *laws* of logic, and the so-called "logical truths," which Frege and Russell regard as instances of these laws, are for him merely sentences that are devoid of sense because they convey no information. Consequently, far from expressing truths that lie at the basis of all inference-licensing implications—justifying their status *as* inference-licensing—the primitive laws of logic are to be viewed as expressing no facts of any kind. Secondly, even if Russell's conception of the nature of logic were correct, even if the laws of logic were not *sinnlos*, the appeal to logical laws in the explanation of logical entailment would nonetheless be *superfluous*. If one proposition entails another, the entailment holds simply in virtue of the propositions' being the very propositions they are. There is no need to invoke any logical framework, so to speak, holding premise and conclusion in place so that the relation of material implication obtaining between them may qualify *also* as a relation of entailment. Something very close to this thought is evident in the "darkly metaphorical" passage quoted earlier:

If the truth of one proposition follows from the truth of others, this expresses itself in the relations in which the forms of these propositions stand to one another, and we do not need to put them in these relations first by connecting them with one another in a proposition; on the contrary, these relations are internal, and exist as soon as, and by the very fact that, the propositions exist. (5.131) (My translation)

As I read it, this passage is claiming that in order to appreciate that q follows from p there is no need to first show that the proposition "if p then q " can be known to hold on the basis of the laws of logic alone. This is so because we do not need to think of the validity

of the inference as owed to the logical truth of the corresponding conditional (which, on the view Wittgenstein is opposing, amounts to this conditional's being derivable from primitive logical laws). Rather, the validity of the inference is owed merely to an internal relation between the forms of the premise and the conclusion. The holding of this relation is not something that can be put into words. It is something that shows itself forth when we recognize the propositions for what they are: "That the truth of one proposition follows from the truth of other propositions, we perceive from the structure of the propositions" (5.13).

This is *relatively* clear, but at 5.11 Wittgenstein muddies the issue by seeming to express a rival conception of logical entailment. He says:

If the truth-grounds which are common to a number of propositions are all also truth-grounds of some one proposition, we say that the truth of this proposition follows from the truth of those propositions.

This remark needs to be handled with some delicacy, since it seems to invite something analogous to the criticism that John Etchemendy has recently levelled against Tarski's analysis of logical consequence.⁵² By focusing on the fact that the truth-grounds of the premises are also truth-grounds of the conclusion, we seem to leave out the fact that the truth of the premises *guarantees* the truth of the conclusion. This idea is at least more nearly conveyed by the thought of 5.132 that relations of entailment just consist in internal relations between the forms of propositions. So one wonders why Wittgenstein should have troubled to formulate this apparently less appealing analysis.

I would conjecture that 5.11 is supposed to function as a guide to the adoption of a perspicuous notation—a notation in which internal relations of form between propositions are transparent. Its antecedent is the kind of thing we find ourselves

mouthings in the attempt to give voice to what we discern when we recognize the internal relatedness of the forms of premise and conclusion. The antecedent of 5.11 is the attempt to give propositional expression to something that can only be shown, and is, strictly speaking, nonsense. But like much of the *Tractatus*, it nonetheless has value in getting us to *do* something. It moves us to adopt a notation that makes visually apparent the internal relations between the forms of propositions. In what we might call the “truth-table notation,” for example—a notation where propositions are expressed in the form of truth-tables (cf. 4.442)—we can visually read off the relations of form from the notation itself, for these relations are expressed by facts about the distribution of “Ts” and “Fs” among the rows of the propositional signs.⁵³ It is because we can think of the rows of a sign in the truth-table notation as corresponding to the unofficial idea of “truth-possibilities,” and the rows on which a “T” is entered as corresponding to the unofficial idea of “truth-grounds,” that we recognize the truth-table notion as a good notation for making clear the internal relatedness of certain propositions.

By framing propositions in a perspicuous notation such as the truth-table notation we make more plausible Wittgenstein’s idea that propositions contain *their own* resources for revealing themselves to stand in the inferential relationships in which they stand (cf. 6.1265). If *in practice* we sometimes need to derive a conditional in an axiom system in order to recognize the corresponding inference as valid, that is only because we are creatures with quite limited logical capacities. According to Wittgenstein, an axiom system provides for what he calls proof *in logic*, which is to say, the derivation of tautologies from tautologies (cf. 6.126). Unlike the proof of one senseful proposition on the basis of another, this kind of proof serves as no more than a “mechanical aid to the recognition of tautology in complicated cases” (6.1262) (my translation). Most

importantly, an axiom system is not to be viewed as systematizing any supposed body of propositionally expressible “logical knowledge.”

The criticisms stated at 5.132 do not, however, exhaust Wittgenstein’s critique. The remark about all propositions of logic being “of equal status” at 6.127 is a further implicit criticism, since it challenges the conception of intrinsically primitive logical laws upon which a non-system-relative conception of entailment rests.⁵⁴ And there is a further criticism, again not explicitly formulated, but clearly available from the Tractarian standpoint. This is the worry that if we were to take derivability of the conditional ‘ $p \supset q$ ’ from primitive laws of logic as a reductive analysis of the logical entailment of q by p , we would risk making the facts of logical entailment seem to rest upon facts that are not, intuitively speaking, logical. For, from Wittgenstein’s point of view, what Frege and Russell *call* the laws of logic are not intuitively logical laws at all. They are merely peculiarly compendious generalizations, which speak about what is actually the case, and this gives them the appearance of especially general empirical laws—laws that Wittgenstein takes to describe features of reality that might have been otherwise (6.111).⁵⁵

[5] *Wittgenstein’s positive view*

When we infer from p to q : “The method of inference is to be gathered from the two propositions alone” (5.132). The “method of inference” is the particular form of argument employed in the inference. It is to be gathered “from the two propositions alone” because it is something which can be conveyed only by being exhibited *in* propositions. The point is a corollary of Wittgenstein’s denial of the expressibility of logical form. He says:

Propositions cannot represent logical form: this mirrors itself in propositions.

That which mirrors itself in language, language cannot represent.

That which expresses *itself* in language, *we* cannot express by means of language.

Propositions *show* the logical form of reality.

They exhibit it. (4.121) (My translation)

These remarks apply to “logical form” in a rather wide sense, but it is clear from the comment on this thought at 4.1211 that Wittgenstein is thinking, among other things, of the inexpressibility of the internal relatedness of the forms of premise and conclusion in a valid argument. He says:

Thus a proposition “*fa*” shows that the object *a* occurs in its sense, two propositions ‘*fa*’ and ‘*ga*’ show that the same object is mentioned in both of them. If two propositions contradict one another, then their structure shows it; *the same is true if one of them follows from the other*. And so on (4.1211). (My emphasis)

As we have seen, Wittgenstein’s positive view is that the inference-licensing relation between the forms of the propositions is an internal one that shows itself in the structures of the propositions. But what does Wittgenstein mean by an “internal relation”? And which view exactly does he mean to be opposing by insisting on the internality of the relation of logical entailment?

Wittgenstein says that a relation is internal if it is unthinkable that it should fail to relate its actual relata (4.123). If one shade of blue is brighter than another, then it is unthinkable that these very shades should not be so related (4.123). Wittgenstein does not explain what it is for a state of affairs to be “thinkable,” but one supposes he must have something non-psychological in mind—perhaps a state of affairs is thinkable in Wittgenstein’s sense just in case there is a proposition that expresses its obtaining. I shall

pursue this idea in more detail later, but first I want to delve a little into the pre-history of Wittgenstein's discussion of the notion of "internality."

In presenting entailment as an internal relation, Wittgenstein is issuing a direct challenge to Russell's view that *all* relations are external. Russell, in turn, adopted this view as a central plank in his rebellion against the monistic idealism of F. H. Bradley. In his 1899 essay "The Classification of Relations" Russell says:⁵⁶

Mr. Bradley has argued much and hotly against the view that relations are ever purely "external". I am not certain whether I understand what he means by this expression, but I think I should be retaining his phraseology if I described my view as the view that *all* relations are external.

Russell's view, notice, is that he does not know what Bradley means, but that he nonetheless regards it as wrong. Notice also the comprehensiveness of Russell's rejection of Bradley. He does not merely assert that *some* relations are external, which is all he needs to say in order to disagree with Bradley; he goes the whole hog and asserts the externality of *all* relations.⁵⁷

The view that all relations are external looks extremely unpromising, if we understand externality in terms of Wittgenstein's notion of "thinkability." For we most naturally view propositions as incapable of standing in relations of logical entailment different from those in which they actually stand. We reason: if a proposition were to stand in relations of logical entailment different from those in which it actually stands, that could only be in virtue of its having a different logical structure (containing different logical constants);⁵⁸ but if a proposition contained different constituents, it would not be *that* proposition.

But it does not seem likely that Russell is taking himself to be committed to the externality of relations in *Wittgenstein's* sense. His view would seem to be, rather, that all relations are “not-internal,” in whatever sense *Bradley* attaches to the term “internal.” For *Bradley*, a relation is “internal”—or in his terminology “intrinsic”—when it “effects” or “passes into” or again, “penetrates the being of” its terms.⁵⁹ How Russell understands this idea can be gathered from his essay “The Monistic Theory of Truth” of 1906-7, where he offers the following characterization of what it means to say that there are *external* relations.⁶⁰

There are such facts as that one object has a certain relation to another...[and such facts] do not imply that the two objects have ... any *intrinsic* property distinguishing them from two objects which do not have the relation in question.

If we take “imply” here to mean something stronger than “materially imply,” we might read Russell as denying something close to what we should today call a supervenience claim. To say that there are external relations is to say that there are relations that do not supervene on the intrinsic properties of their relata. The idea Russell is gesturing at might be expressed, using the apparatus of possible worlds, as follows:

A relation *R* is external (in the sense of not penetrating the being of its terms) just in case there are possible worlds *w* and *w*¹ and terms *X*₁, *Y*₁, *X*₂, *Y*₂ such that *X*₁ and *Y*₁ are in *w* and *X*₂ and *Y*₂ are in *w*¹, neither the *X*'s nor the *Y*'s differ intrinsically, and the relation relates only one of the pairs of terms.⁶¹

One might well have reservations about glossing the remark in this way, for the formulation I have given employs modal notions for which, officially at least, Russell had

no time.⁶² But, we should bear in mind that, in practice, Russell does not consistently maintain his antimodal stance. He is prepared to invoke modal notions both in 1902, in speaking of logico-mathematical truths as governing “not only the actual world but every possible world”;⁶³ and also in 1905 in characterizing—though not defining—propositions, as *possible* objects of belief.⁶⁴ It does not seem outlandish, therefore, to suppose that Russell might have been employing modal notions, tacitly and unofficially, in trying to make sense of the idea that an internal relation “penetrates the being of its terms.”

Let us suppose, then, that Russell’s talk of externality is to be understood in terms of non-supervenience, and his talk of internality in terms of supervenience. Is this proposal compatible with his claim that *all* relations are external? It seems not. For consider what it would be to assert the externality of, for example, the logical entailment relation. One would have to claim that there are two worlds w and w^1 and two pairs of propositions $\langle P_1, Q_1 \rangle$ and $\langle P_2, Q_2 \rangle$ such that the first pair is in w and the second in w^1 , both the P ’s and the Q ’s agree in all their intrinsic properties, and the members of only one pair stand related by logical entailment. But this is plainly false, for the terms in question, being propositions, are items for which a lack of intrinsic difference amounts to a lack of numerical distinctness. If two propositions share all their intrinsic properties, they must contain exactly the same constituents, and must combine these constituents in precisely the same way; but then they will just be the *same* proposition.⁶⁵ So to suppose that the relation relates one pair of propositions at one world, and fails to relate an intrinsically identical pair at another, is just to suppose it equally possible (hence equally thinkable) that the relation of entailment should either relate or not relate one and the same pair of propositions. But, that was the very position we were trying to avoid by pursuing the detour by way of Bradley’s conception of an internal relation. (We pursued

the detour in the belief that not even Russell would wish to allow that a proposition could be conceived as having a range of *different* conceivable positions in the inferential network.) So Russell must, in the end, deny that there are any such worlds as the ones we have tried to describe; and he must, accordingly, judge the relation of logical entailment to be internal.

There are two very different kinds of conclusion we might draw from these reflections. On the one hand, we might say that once the thesis that all relations are external has been made clear—by giving a precise sense to the notion of externality—it is found to be false. But, alternatively, we might say that because he must reject our construal of the notion of internality Russell has in fact failed to attach any meaning to the expression “external,” and so has failed to invest the sentence “All relations are external” with sense.

Russell seems to want to occupy an impossible space between these two positions. He isn't sure what Bradley means by calling a relation “internal,” but he is confident that Bradley must mean *something*, and certain that, whatever that should turn out to be, the truth is that there are no internal relations *in that very sense*. In effect, Russell keeps his thesis viable by deferring scrutiny of its key notions. It is as though he takes himself to see matters clearly enough to know that Bradley is wrong, but not sufficiently clearly to know *what* he is wrong about.

The *Tractatus* contains a strand of thought that is deeply critical of this philosophical attitude. Where Russell would take himself to have propounded a substantive, defensible thesis—albeit one requiring clarification—Wittgenstein would

take him to have failed to attach meaning to certain of his words, and so to have uttered nonsense—albeit nonsense with a strong tendency to pass itself off as sense (cf. 5.4733).⁶⁶

Given that such a style of criticism is so thoroughly Tractarian in spirit, and so plausibly applicable to the case at hand, one might wonder why Wittgenstein refrains from applying it here. Instead of saying that the question whether all relations are external makes no sense because some terms employed in its formulation lack a meaning, he seems to propose a way of making sense of it. That is to say, he appears to give his own characterization of what it means for a relation to be internal, and to claim that the relation that justifies inference is internal in just this sense. Is he, then, disagreeing with Russell in the same spirit in which Russell disagreed with Bradley? Is he claiming to have discovered the core of what Bradley and Russell must have meant by their talk of “internal relations,” and to have a counterexample to Russell’s thesis that all relations are external? One might be forgiven for thinking so, for, having drawn the distinction between external relations, which can be put into words, and internal relations, which show themselves in language, Wittgenstein writes: “Now this takes care of the vexed question ‘whether all relations are internal or external’” (4.1251) (My translation.)

However, the impression that Wittgenstein regards himself as having “taken care of” the question by *solving* it, begins to evaporate as soon as one scrutinizes the notion of “thinkability” appealed to in Wittgenstein’s exposition of internality. Earlier we made do with the following suggestion: To say that a state of affairs (that p) is thinkable is just to say that there is a proposition which says that p. But this proposal needs refining,⁶⁷ for we shall have to decide whether, for the purposes of this characterization, tautology and contradiction count as “propositions.” And here we seem to be in a bind. If we say they do, we commit ourselves to the thinkability of contradictions. If we say they don’t, we

commit ourselves to counting the circumstance that either it is raining or is not, as *unthinkable*. But our intuitive sense is that this is something we *can* think. After all, it is a *logical consequence* of something we can think, namely that it is raining.

The best way to respond to this dilemma is to blunt one of its horns. We need to insist that it is not, strictly speaking, *thinkable* that either it is raining or it is not raining. To make this ruling plausible we shall have to deny that the technical notion of “thinkability” in play here has very much to do with the intuitive notion of what we can think. Since tautologies do not express “thoughts” in Wittgenstein’s technical sense of the term—that is, *senseful* propositions (4), what they do express, if anything, is not in this sense thinkable.⁶⁸ Taking such a position would allow us to reformulate the explanation of thinkability by substituting “proposition with sense” for “proposition” in the formulation given above. This would yield the mildly satisfying result that the circumstance that *p* is *thinkable* just in case there is a *thought* (that is, a senseful proposition) which says that *p*.

But how are we to understand this quantification over thoughts? On the face of it, we are saying that when the state of affairs that *p* is thinkable, for some *q*, *q* is a proposition, *q* has sense and *q* says that *p*, where “*q*” is a variable which occupies an argument place accessible to names of propositions.⁶⁹ We can make sense of this statement only if we can make sense of such sentences as “*a* has sense,” where “*a*” is a singular term designating a proposition. But because propositions, for Wittgenstein, are facts (cf. 3.12 and 3.14), and because facts cannot be named (cf. 3.144),⁷⁰ we will not be able to treat these singular terms as Tractarian names. It seems that we will need to treat sentences containing singular terms that purport to designate propositions, as admitting of paraphrase by sentences in which no singular terms purporting to designate propositions

occur.⁷¹ However, Wittgenstein provides no proposal that would enable us to pull off this trick. We cannot employ a combination of the description theory of (apparent) names and the theory of descriptions to eliminate the apparent name for a proposition, since, even if we could find a satisfactory description, the resulting application of the theory of descriptions would re-introduce quantification over propositions. Nor can we apply the *Notebooks*'s analysis of a sentence predicating a property of a complex: namely, $\phi a. \phi b. aRb = \text{Def } \phi[aRb]$ ⁷² (cf. 3.24), for whatever its prospects more generally, the proposal will obviously fail in connection with the kinds of things one will want to say of propositions. (Consider, for example, the result of substituting 'has sense' or 'is a proposition' for ' ϕ ' in the above definition.) In the end, then, Wittgenstein's talk of thinkability and his explanation of 'internality' turn out to be in the same boat as Russell's talk of 'internality.' In both cases we have promisory notes, but no satisfactorily completed explications.

Why, then, does Wittgenstein trouble to half formulate the notion of an "internal relation"? Why does he act as though there is a well-defined question whether all relations are external, upon which to take sides? In this case I believe there may be something to the idea of Cora Diamond that for certain heuristic purposes Wittgenstein sometimes behaves as though certain notions to which he has (as yet)⁷³ attached no meaning make full sense. In the present case, the temporary adoption of this stance at very least has a certain diagnostic value: it enables us to detect various gradations of philosophical error, and to see why someone might have taken claims involving internal relations to make sense.

Consider, for example, a paradigmatic report of the obtaining of a supposedly internal relation: “This shade of blue is darker than that one.” I take it that the idea behind Wittgenstein’s insistence on the “bi-polarity” of the proposition in his pre-Tractatus writings⁷⁴ is that one who took this sentence to make sense could be brought to realise that it fails to say anything by being invited to think through what would be said by its negation. On realizing that there is nothing they succeed in thinking when they attempt this exercise, they will come to see that the original unnegated sentence fails to exclude any coherent state of affairs, and so, after all, fails to express a thought.

Wittgenstein’s diagnosis would be that the sentence says nothing because we have failed to attach any significance to the fact that ‘is darker than’ stands between names of *shades* of colour (cf. 5.473 & 5.4733). He would add that we are inclined not to notice this because we tend to construe the sentence on the model of the fully senseful sentence: “This patch of cloth is darker than that one,” a sentence which contains an expression for a proper external relation. (In so doing, we will have missed the subtle “shift” from a genuine external relation to a “formal,” or “pseudo-relation” (cf. 4.123)). We may conclude that Wittgenstein has identified a class of utterances that express no thoughts, even though it is quite natural to take them as expressing thoughts, namely, those utterances to which we are inclined to prefix the phrase: “It is unthinkable that it should not be the case that....”

The notion of an internal relation, which presupposes the notion of thinkability, may also play a role at an earlier stage in our considerations. By temporarily availing oneself of the notion of an internal relation, one may arrive at a more refined—if still ultimately problematic—view of *which* internal relations there are. One might, for example, be persuaded that the only internal relations are ones that hold or fail to hold of

propositions. Wittgenstein's remark that the only necessity is logical necessity (6.37, cf. 6.375) might be taken to suggest such a view, for it invites us to construe talk of internal relations between *objects* as the expression of an inchoate insight into the obtaining of internal relations between *propositions*. So, for example, while G. E. Moore says that a complex stands in an internal relation to its constituent part,⁷⁵ Wittgenstein says: "A proposition about a complex stands in internal relation to the proposition about its constituent part" (3.24). (The relation in question might be the one "shown forth" by the *Notebooks*'s analysis mentioned above.)

I take it that this unofficial view of which internal relations there are, is supposed to play an heuristic role in guiding the project of analysis. Suppose that we help ourselves temporarily to the notion of an internal relation. If we then acquiesce in the *Tractatus*'s view of which internal relations there are—that is, if we come to believe that whatever we formerly regarded as a conceptual connection (e.g., that between being coloured and being extended, or being known and being true) is, in fact, an internal relation of form (that is, a logical entailment) holding between *propositions*, we will be led to try to analyse all propositions into a form where these connections become fully apparent, and, for Wittgenstein, this will be a form in which propositions reveal themselves to be truth-functions of elementary propositions.

Thus, by taking a stand on which internal relations there are, we motivate and direct the project of analysis. Once this conception of analysis is accepted and incorporated into our philosophical practice, our talk of "internal relations" can be jettisoned as part of the ladder we are to throw away (cf. 6.54).

Notes

¹ My thanks to: Ed Holland, Ori Simchen, Peter Sullivan, and Markus Stepanians, Jon Curtis, Stephen Everson, Steven Gross, Robin Jeshion, Oystein Linnebo, Michael Potter, Peter Railton, and Jason Stanley. I owe a special debt of gratitude to Warren Goldfarb, Richard Heck, Charles Parsons, and Jamie Tappenden, each of whom provided generous comments on numerous drafts. Various incarnations of the paper were given as talks at Harvard University, Cornell University and the University of Michigan. The discussion on each occasion led to extensive revisions. My thanks to all who participated.

²Ludwig Wittgenstein, *Logisch-Philosophische Abhandlung: kritische Edition*, B. McGuinness and J. Schulte eds. (Frankfurt am Main: Suhrkamp, 1989). I have followed one or the other of the following translations, except where indicated. *Tractatus Logico-Philosophicus*, trans. C. K. Ogden (London: Routledge and Kegan Paul Ltd, 1922). *Tractatus Logico-Philosophicus*, trans. D. F. Pears and B. McGuinness (London: Routledge and Kegan Paul Ltd., 1961).

³The German contains quotation-marks that are missing from Ogden's translation.

⁴See, for example: Gordon Baker, *Wittgenstein, Frege and the Vienna Circle* (Oxford: Blackwell, 1988), p. 130; Anthony Kenny, *Wittgenstein*, first published 1973 (Harmondsworth: Penguin Books Ltd.), p. 98; H. O. Mounce, *Wittgenstein's Tractatus* (Oxford, Blackwell, 1981), p. 46.

⁵"Foundations of Geometry: First Series," p. 319, in *Collected Papers*, Brian McGuinness, ed., trans. Hans Kaal et. al. (Oxford: Blackwell, 1980), pp. 273–284.

⁶*Foundations of Logic 1903-05, The Collected Papers of Bertrand Russell, Vol. IV*, edited by Alasdair Urquhart with the assistance of Albert C. Lewis (London: Routledge, 1994), p. 515. A point that we should keep in mind throughout our discussion is that often—though, as we shall see, by no means *always*—Russell uses the word “implies” to express the relation of *material* implication, rather than logical entailment.

⁷These passages don't *exclude* inference rules from the class of ‘laws of inference,’ and I know of no other that does.

⁸Thomas Ricketts, “Frege, *the Tractatus*, and the Logocentric Predicament,” *Noûs* (1985), pp. 3–15.

⁹See Ricketts, op. cit., p. 7: “Unlike Russell, Frege is absolutely clear on the difference between logical laws and inference rules, as well as the need for both in his axiomatic formulation of logic.” Ricketts cites no texts to support his implication that Russell was not clear on the difference.

¹⁰“A law of inference, a proposition in its own right, could justify an inference only by serving as a premise from which a conclusion is drawn. How else?” Ricketts, op. cit., p. 12.

¹¹Carroll, L. “What the Tortoise said to Achilles,” *Mind* 4 (1895), pp. 278–80.

¹²*Principia Mathematica* to *56, with A.N. Whitehead, first published 1910–13 (Cambridge: Cambridge University Press, 1990), part I, p.106.

¹³Ibid.

¹⁴Often Russell employs as rules principles of deduction that occur elsewhere as premises simply in order to avoid the tedium of providing fully rigorous proofs.

¹⁵*The Principles of Mathematics* (London: Allen & Unwin, 1903), Part I, chapter 3, section 45.

¹⁶It is important to note that Russell does not himself regard what is “formally necessary” as necessary. Here he means by “formally necessary” what would *seem* to be necessary from the point of view of an over-zealous formalizer— i.e. one who wishes the formalism never to break down.

¹⁷Russell in fact overshoots his target, failing to record substitution even as an inference rule in *Principia*. (For Russell’s later acknowledgement of this oversight see *Introduction to Mathematical Philosophy*, p. 151, n. 1.)

¹⁸*Principles* § 18.

¹⁹Op. cit. p. 132.

²⁰I say “*may* need to be” since, obviously, there will be some inferences, such as an inference by *modus ponens*, that will not require such mediation.

²¹For an illuminating discussion of this conception of logic see Warren Goldfarb, “Frege’s Conception of Logic,” in J. Floyd and S. Shieh, eds., *Futures Past: Reflections on the History and Nature of Analytic Philosophy* (Cambridge, Mass.: Harvard University Press, 2001). Ricketts gives a closely parallel account in “Pictures, logic, and the limits of sense in Wittgenstein’s *Tractatus*,” in Sluga and Stern eds., *The Cambridge Companion to Wittgenstein* (Cambridge: Cambridge University Press, 1996), pp. 59–99.

²²See, for example, “Foundations of Geometry: Second Series”, pp. 387 & 425, “Negation” p.145, “Compound Thoughts” p. 47 (All essays are reprinted in Frege *Collected Papers*, Brian McGuinness, ed., trans. Hans Kaal et al., Oxford Blackwell, 1984). Faced with the objection that at some level this misrepresents mathematical practice, which includes proof by contradiction, Frege offers an alternative story of what is really going on. The objection is raised: “Surely one can make deductions from certain thoughts purely hypothetically without adjudging the truth of the latter.” Frege replies: “Certainly, purely hypothetically! But then it is not these thoughts that are the premises of such inferences. Rather, the premises are certain hypothetical thoughts that contain the thought in question as antecedents.” (“Foundations of Geometry: Second Series” p. 425). A fuller explanation along the same lines is contained in “Logic in Mathematics” in

Gottlob Frege: Posthumous Writings, Hans Hermes, Friedrich Kambartel and Friedrich Kaulbach, eds., trans. Peter Long and Roger White (Oxford: Blackwell, 1979), pp. 245 ff.

Paolo Mancosu has made clear that Frege's view on this point is not idiosyncratic. Frege appears to be following in the footsteps of Bolzano, who held proof by contradiction to be always eliminable in favour of direct proof from true axioms. See Paolo Mancosu, *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*, Oxford: Oxford University Press, 1996, ch. 4. (I am grateful to Jamie Tappenden for bringing Mancosu's work to my attention.)

That Russell regarded a genuine inference from a false proposition as an impossibility is evident from his characterization of the "therefore" relation, which is the relation that is supposed to hold between premise and conclusion when we infer the latter from the former. Russell says that the [holding of] the relation implies that hypothesis [i.e., premise] and consequent [i.e., conclusion] are true. See *Principles* § 15.

²³By "logical entailment" I mean the converse of the relation of logical consequence, but understood in such a way that propositions as well as sentences can properly be said to stand in this relation. (My aim is to have a neutral term for a relation understood to have in its field items that are the bearers of truth and falsity, whatever they may turn out to be.)

²⁴This formulation is suggested by some remarks in Wittgenstein's Cambridge lectures of 1931. In the course of some reflections on logical conceptions from the *Tractatus*, Wittgenstein remarks: "Inference is justified by an internal relation which we see; the only justification of our transition is our looking at the two terms and seeing the internal relation between them" and a moment later he continues: "Propositions do not follow from one another as such; they simply are what they are" (*Wittgenstein's Lectures Cambridge, 1930–32, From the notes of John King and Desmond Lee*, Desmond Lee, ed., first published 1980, Midway Reprint (Chicago: University of Chicago Press, 1989), p. 57).

²⁵G. E. Moore, "Necessity," *Mind* n.s. vol. 9, no 35, July 1900, pp. 289–304.

²⁶*Foundations of Logic 1903–05, The Collected Papers of Bertrand Russell, Vol. IV*, edited by Alasdair Urquhart with the assistance of Albert C. Lewis (London: Routledge, 1994), p. 513.

²⁷Ibid.

²⁸Here, once more, Russell is using the expression "laws of logic" for the *axioms* of his system. If he did not mean *only* the axioms, there would be no need to distinguish those cases in which the relation of premise to conclusion is "contemplated" by the laws of logic from those in which it is deducible from them.

²⁹Here I am simplifying Russell's account by treating deducibility from the laws of logic as equivalent to derivability from them. These are not strictly the same notions, but it is easy to show that they are extensionally equivalent.

³⁰It is not known whether Wittgenstein read Russell's 1905 paper, but it is quite possible that he should have learned of Russell's views on deducibility during his numerous conversations with him at Cambridge during the years 1911–13. These conversations were long and involved, and the topic was usually the nature of logic.

³¹*Principia*, section A, p. 90.

³²See *Principles* §§15 and 38.

³³Perhaps because of his awareness of Russell's own doubts about self-evidence as a criterion of the logically primitive—doubts which arose after Russell's discovery that the reduction of arithmetic to logic requires the axiom of reducibility—Wittgenstein is more cagey about directly attributing this view to Russell than to Frege (The attribution to Frege is made at 6.1271.). *Tractatus* 5.4731, however, suggests that Wittgenstein believed that Russell's most considered view would have been to maintain a tight connection between self-evidence and the primitively logical. Wittgenstein writes: "Self-evidence, of which Russell has said so much, can only be discarded in logic by language itself preventing every logical mistake. That logic *a priori* consists in the fact that we *cannot* think illogically." The last sentence strongly suggests that Wittgenstein took himself to be replacing Russell's psychological criterion of primitive logicality (*viz.* obviousness) with a non-psychological notion: unthinkability of the negation. I will have more to say about the notion of "unthinkability" later.

³⁴I am grateful to Warren Goldfarb for stressing the importance of this passage in this connection.

³⁵"Foundations of Geometry: Second Series," pp. 423–4.

³⁶The description of the account as "proof-theoretic" needs to be heavily qualified. Frege, of course, is speaking of thoughts, not uninterpreted symbolisms; so his account is by no means a contribution to "proof-theory" in the sense familiar to contemporary philosophers—*i.e.*, a theory that deals with syntactic objects known as "proofs."

³⁷*Ibid.*, p. 429.

³⁸An illuminating discussion of this point is contained in Thomas Ricketts's paper: "Frege's 1906 Foray into Metalogic," *Philosophical Topics* (Fall 1997), pp. 169–188.

³⁹See Tyler Burge, "Frege on Knowing the Foundation," *Mind*, 107, no 426 (1998), pp. 305–47, and Robin Jeshion, "Frege's Notions of Self-Evidence" (*Mind*). I am grateful to Jamie Tappenden for bringing Jeshion's paper to my attention, and to Robin Jeshion for allowing me to discuss a pre-publication version.

⁴⁰Whether a truth has this feature will depend on whether the truth has the generality required for one to be able to derive other propositions from it. (Instances of the law of identity, for example, will not.)

⁴¹Jeshion's story is actually a little more complicated than this, for it takes the quality of being self-supporting as a necessary but not sufficient condition for being a basic truth. (Basicness additionally requires that the simple constituents of the truth are themselves 'basic' in another sense.) The details, however, are less important for our purposes than the idea that Frege has some notion of an objectively basic truth.

⁴²It is worth mentioning, in support of Burge and Jeshion, that Frege's conception of an *objective* structure of justification is not unique among Nineteenth Century views on the foundations of mathematics. Paolo Mancosu (op. cit., pp. 102–3) quotes a passage from Bolzano's 1810 publication *Contributions to a better-founded presentation of mathematics*, which contains a striking anticipation of this idea:

Now the further question arises, what should properly be understood by the *proof* of a truth? One often calls every sequence of judgments and inferences by which the truth of a certain proposition is made generally *recognizable and clear*, a *proof of the proposition*. In this *widest* sense, all true propositions, of whatever kind they may be, can be proved. We must therefore take the word in a *narrower* sense and by the *scientific proof* of a truth we understand the representation of the *objective dependence* of it on *other truths*, i.e. the derivation of it from such truths which are to be considered as the *ground for it*—not fortuitously [*nicht Zufälliger Weise*]—but in *themselves and necessarily*, while the truth itself must be considered as their *consequence*. (*Beiträge zu eine begründeteren Darstellung der Mathematik, Erste Lieferung* (Prag. 1810), §12)

⁴³Burge argues that the nonpsychological notion of self-evidence applies to inference rules as much as to laws of logic (Burge, op. cit., p. 322.).

⁴⁴Ricketts (op. cit.) claims that Frege was reluctant to endorse the strategy he sketches in "Foundations of Geometry: Second Series" for giving independence proofs, and attributes this reluctance to Frege's awareness that he lacked precise characterizations of the notions of 'logical law' and 'logical inference'. Since such characterizations could be developed from characterizations of 'primitive logical law' and 'primitively logical inference,' I take it that Ricketts would read Frege as similarly pessimistic about the clarification of these notions.

⁴⁵The standard translations by Ogden, and by Pears and McGuinness, are misleading here. Each suppresses the reference to Fregean-sounding "basic laws".

⁴⁶The German is "*als Kriterium*." By omitting the article Wittgenstein leaves it unclear whether he takes Frege's view to be that psychological obviousness is *the* criterion (i.e., either a necessary and sufficient condition, or, possibly, a sufficient condition) or merely *a* criterion (i.e., either a necessary condition, or, possibly, a defeasible "way of telling"). The context of the remark suggests that he took Frege to regard obviousness as at the very least a necessary condition of being an essentially basic logical law.

⁴⁷At 6.1271 Wittgenstein speaks of the degree of self-evidence as supposed to be the criterion of a “logical proposition,” not of a “primitive logical proposition,” but it seems likely that he has the latter notion in mind here, for the remark begins with Wittgenstein’s claim that Frege was wrong to think the number of *basic* laws non-arbitrary, since one could develop logic from one basic law, namely, the conjunction of Frege’s basic laws. Wittgenstein then envisages Frege as challenging the status claimed for this conjunction—i.e. of being a primitive proposition—on the ground that it lacks self-evidence. Moreover, we have seen that Frege and Russell both sometimes speak of the “logical” when they mean the “primitively logical,” so it would be no surprise if Wittgenstein also had developed this loose way of speaking.

⁴⁸See Brian McGuinness *Wittgenstein a Life: Young Ludwig (1889–1921)* (London: Penguin Books, 1988), chaps. 4–6.

⁴⁹*Grundlagen* § 17. I am grateful to Jamie Tappenden for bringing this passage to my attention.

⁵⁰For the significance of the scare-quotes see footnote 36 above.

⁵¹As evidence that the Wittgenstein of the *Tractatus* had read this work—besides his acknowledgement of the “great works of Frege” in the Preface of the *Tractatus*—we may cite *Tractatus* 3.3, which loudly echoes the *Grundlagen*’s famous context principle.

⁵²John Etchemendy *The Concept of Logical Consequence*, (Cambridge, Mass.: Harvard University Press, 1990), chap. 6.

⁵³At 6.122 Wittgenstein emphasizes this visual aspect of inference:

...we can get on without logical propositions, since we can recognize in a suitable notation the formal properties of the propositions by mere inspection (*durch das blosse Ansehen*).

And there are many other passages that stress this visual theme (see, e.g., 6.1221). It seems that Wittgenstein is envisaging an adequate notation to be one in which the necessity of the relation of entailment is expressed by visual means. That one *cannot* draw a quincunx without drawing four triangles is visually apparent in the figure itself. I would conjecture that an adequate notation is one that is supposed in a similar way to render visually apparent the necessity of the relation of logical entailment. (I am indebted to Peter Railton for this point.)

⁵⁴The notion of psychological self-evidence might also yield a non-system relative notion of derivability, but, as we have noted, it too is a notion to which Wittgenstein takes exception (6.1271)

⁵⁵To require that the recognition of logical inferential relations should not turn on the discernment of any worldly facts is not to embrace a conception of logic as somehow entirely without existential presuppositions, a free-floating entity having its nature quite

independently of whether there is a world. As 5.5521 makes clear, such a conception would render mysterious how such a logic could apply to the world: “[If logic were prior not just to how the world is but to the fact of its very existence] how could we apply logic? We could say: if there were a logic, even if there were no world, how then could there be a logic, since there is a world?”

For Wittgenstein, logic is simply a by-product of language designed to represent the world. If there were no world to be represented (whether truly or falsely) then there would be no logic either, because there would be no language. “The logical propositions describe the scaffolding of the world, or rather they present it. They “treat” of nothing. They presuppose that names have meaning, and that elementary propositions have sense. And this is their connection with the world.” (6.124)

⁵⁶Bertrand Russell *Philosophical Papers 1896–99, The Collected Papers of Bertrand Russell, Vol. II*, Edited by Nicholas Griffin, Albert C. Lewis, and William C. Stratton (London: Routledge, 1990), p.143. Given the date of this remark, it is worth mentioning that the bulk of the *Principles* appeared in the draft of May 1901, and that most of it was written during 1900. See the Introduction to the 2nd Edition of the *Principles* p. v, and *Toward the “Principles of Mathematics,” 1900–02, The Collected Papers of Bertrand Russell, Vol. III*, Gregory H. Moore, ed., (London: Routledge, 1993), ‘Introduction’ and ‘Chronology’.

⁵⁷Because Russell may still be found speaking of the problems inherent in “the view that relations are not purely external” as late as 1905 (Russell, *Collected Papers Vol. IV.*, p. 502), it is plausible to suppose that he continued to subscribe to the contrary of Bradley’s position throughout the period 1900–1905. Against this, it should be mentioned that Russell does once make a passing reference to internal relations in the *Principles* (§ 412). However, it is far from clear that ‘internality’ in this context is supposed to amount to Bradley’s notion. Russell may merely mean that a relation is ‘internal’ to a progression in the sense that the progression is ordered by that relation, and so in a sense is built up out of it.

⁵⁸I would take the intuitionist interpretation of the logical connectives to amount, in a Russellian setting, to an alternative view of which connectives the proposition contained, not to an alternative view of the proposition’s logical relations.

⁵⁹F. H. Bradley *Appearance and Reality: a Metaphysical Essay*, second edition (Macmillan, 1908), p. 364.

⁶⁰*Philosophical Essays*, first published 1910 (London: Routledge, 1994), pp. 139–140. There is a confusion over the title of this paper that seems worth clearing up. The chapter of *Philosophical Essays* from which this remark is drawn is entitled: “The Monistic Theory of Truth.” The material of the chapter comprises the first two sections of a three-section paper, which Russell delivered to the Aristotelian Society on December 3rd, 1906. The paper was originally entitled “On the Nature of Truth.” The editorial note in *Philosophical Essays* (p. 131), however, incorrectly cites it as: “The Nature of Truth.” See “On the Nature of Truth,” *Proceedings of the Aristotelian Society 1906–7*, New

Series—Vol. 7 (1907), pp. 28–49.

⁶¹We might be tempted to say: “just in case it is possible that X_1 and Y_1 should stand related by R and X_2 and Y_2 not do so, when neither the X 's nor the Y 's differ intrinsically” but this formulation counts as internal, relations that we would intuitively consider external. For the definition only considers pairs of terms contained in the same world. However, a relation that related one pair of items at one world, but failed to relate another intrinsically identical pair at another world is one we should intuitively wish to call external. The point is due to David Lewis. See his *On the Plurality of Worlds* (Oxford: Blackwell, 1986), pp. 14–17.

⁶²The formulation also presupposes the availability of a workable distinction between intrinsic and relational properties; and that distinction is notoriously hard to formulate.

⁶³See the essay “The Study of Mathematics” collected in *Mysticism and Logic*, which Russell wrote in 1902, first published 1917 (Totowa, New Jersey: Barnes and Noble, 1981), p. 55.

⁶⁴See Russell's *Collected Papers Vol. IV*, p. 495.

⁶⁵We might argue for the view that there cannot be two distinct propositions with the same intrinsic properties. Suppose there are two such. Then what rendered them distinct would have to be their occupying distinct positions in a network of inferential relations, for there would be nothing beyond their relational properties to which we could appeal to ground their distinctness. But *that* would not be possible without an intrinsic difference in the propositions. Intrinsically identical propositions must have the same inferential relations to other propositions (though the converse need not hold).

⁶⁶The conception of nonsense I am alluding to here is one that has received considerable illumination from the writings of Cora Diamond. See her *The Realistic Spirit: Wittgenstein, Philosophy and the Mind* (Cambridge, Mass.: MIT Press, 1991), especially ch. 3.

⁶⁷I am grateful to Jamie Tappenden for getting me to re-think this issue.

⁶⁸If they express anything, it is only by showing it forth. What they show is “the logic of the world” (6.22).

⁶⁹Wittgenstein's own account differs from this one insofar as it treats existential quantifications as (in effect) analyzable as (possibly infinite) disjunctions. However, I will continue to use this proposal for purposes of illustration, for the problems it raises are also problems for Wittgenstein's account.

⁷⁰Compare these remarks from the *Notes on Logic*: “Facts cannot be named” (*Notebooks* p. 107), “Frege said ‘propositions are names’; Russell said ‘propositions correspond to complexes.’ Both are false; and especially false is the statement ‘propositions are names of complexes’” (*Notebooks* p. 97).

⁷¹It is clear that in the *Notes on Logic* Wittgenstein envisaged paraphrastic analysis as necessary to eliminate what he called, following Russell, “incomplete symbols” (*unvollständige Zeichen*). These include such phrases as “the sense of p ” and “the meaning of p ” (See *Notebooks* p. 102. And in the same connection see Wittgenstein’s quotation (of Russell’s quotation) of the original German text of the remark from the “manuscript” section of *Notes on Logic*, and Wittgenstein’s comments upon it, in Wittgenstein’s second letter of November 1913 (*Ludwig Wittgenstein: Cambridge Letters, Correspondence with Russell, Keynes, Moore, Ramsey and Sraffa*, edited by Brian McGuinness and Georg Henrik von Wright (Oxford: Blackwell, 1995), p. 50)). The first letter of that month makes clear that Wittgenstein envisaged a similar “analysis” of (contexts containing) the word “fact” (See *Cambridge Letters* p.48). To cement the connection with paraphrastic analysis we should note that the phrase “incomplete symbol” is Russell’s. In chapter III of the Introduction to *Principia*—a chapter entitled “Incomplete Symbols”—Russell and Whitehead characterize the notion thus: “By an ‘incomplete symbol’ we mean a symbol which is not supposed to have any meaning in isolation, but is only defined in certain contexts” (*Principia* p. 67).

⁷²*Notebooks* p. 4. Note that Wittgenstein gives definitions in the reverse of their now conventional order.

⁷³ It remains a question for further inquiry whether in adopting this stance Wittgenstein abandons the hope that signs to which he has given no meaning in isolation might later be given a meaning through an unforeseen contextual definition.

⁷⁴In the 1913 *Notes on Logic* Wittgenstein says that every genuine proposition is “bi-polar” in the sense of being “essentially true-false” (*Notebooks*, p. 98). He means that every proposition is, by its very nature, both capable of truth *and* capable of falsity. He makes clear that bi-polarity also involves the idea that to understand a proposition I must know, and have a full conception of what would have to be the case for it to be true *and* what would be the case for it to be false (*Notebooks*, pp. 94–95 and 98–99). Although bi-polarity is not explicitly mentioned in the *Tractatus*, Wittgenstein is unlikely to have changed his mind about it. He mentions the requirement as late as the *Notebooks* entry for the 2nd of June, 1915 (*Notebooks*, p. 53). If McGuinness is right, this would have been only a matter of months before he set about writing the first seventy sides of the *Prototractatus*, which differs from the *Tractatus* only in minor respects. See his “Wittgenstein’s pre-*Tractatus* manuscripts” in *Grazer Philosophische Studien* 33 (1989), pp. 35–47.

⁷⁵Moore held the relation of a (spatially extended) whole to its part—but not of the part to the whole—to be an internal relation. See “External and Internal Relations” in *G. E. Moore: Selected Writings*, Thomas Baldwin, ed. (London: Routledge, 1993), p. 88. (Originally published in *Philosophical Studies*, London: Routledge and Kegan Paul, 1922, pp. 276–309.)