Answers to Practice Test 4

Proofs by Induction. Use your blue books.

8. $\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$

Base Case. Let n = 1. It suffices to prove that $1 \cdot 1! = (1 + 1)! - 1$

$1 \cdot 1! = 1 \cdot 1$
$= 1$
$= 2 - 1$
$= 2! - 1$
$= (1 + 1)! - 1$

Inductive Case.
Inductive hypothesis: the sum from 1 to k = (k + 1)! - 1
It suffices to show that the sum from 1 to k + 1 = (k + 2)! - 1.
Sum from 1 to k + 1 = sum from 1 to k + (k+1)(k+1)!
= (k + 1)! - 1 + (k+1)(k+1)!, by IH.
= $(k + 2) \cdot (k+1)! - 1$

9. Consider the formal language consisting of vocabulary items 📖 and ⌛. Grammatical strings are defined recursively as follows:

(1) List(📖) is a string.
(2) If 𝒜 is a string, then List(📖)^𝒜^List(⌛) is a string.
(3) There are no other strings.

Prove that in any string, the number of 📖 's is greater than the number of ⌛'s.

First, we use recursionn to define two functions, the number of k's function (n) and the number of ='s function (q):

n(List(k)) = 1
n(List(k)^𝒜^List(=)) = n(𝒜) + 1

q(List(k)) = 0
q(List(k)^𝒜^List(=)) = q(𝒜) + 1

Let φ be an arbitrary string
Base Case. String φ = List(k).
n(φ) = 1, q(φ) = 0, and 1 > 0.

Inductive case. φ = List(k)^ψ^List(=).
Inductive hypothesis: n(ψ) > q(ψ).
n(φ) = n(ψ) + 1, and q(φ) = q(ψ) + 1,
So n(φ) > q(φ).

Bonus Problem. (10 points maximum)

Use the following recursive definition of the concatenation function on lists :

1. If x is a list, then x^NIL = x.

2. If x and y are lists, then x^Cons(y,a) = Cons(x^y, a).

Prove that for all lists x, y and z, (x^y)^z = x^(y^z). [Hint: prove by induction on the length of z.]

Base Case. Let z = NIL. Show for all x, y, (x^y)^NIL = x^(y^NIL).
(x^y) ^ NIL = (x ^ y).
= x^ (y ^ NIL), since y = y ^ NIL.

Inductive Case. Let z = Cons(w,a).
Inductive hypothesis: for all x, y, (x^y)^w = x^(y^w).
Show: for all x, y, (x^y) ^ z = x^(y ^ z),
(x^y) ^ z = (x^y) ^ Cons(w,a)
= Cons(((x^y)^w),a), by definition of ^
= Cons((x^(y^w)),a), by IH.
= (x^Cons((y^w),q), by definition of ^.
= (x^(y^Cons(w,a)), by definition of ^.
= (x^(y^z))