Phl 313K

Fall ’02

Test 3 Proof solutions

A. Prove that **$(a - (b \cap c)) = (a - b) \cup (a - c)$**

We will use Extensionality. First, we must prove that every member of **$(a - (b \cap c))$ is** a member of **$(a - b) \cup (a - c)$,** and vice versa. Let **d** be an arbitrary object.

($\rightarrow$) Assume $d \in$ **$(a - (b \cap c))$**. Then $d \in a$ and $d \notin (b \cap c)$. So, $\neg(d \in b \ \& \ d \in c)$. By De Morgans law, it follows that $d \notin b$ or $d \notin c$.

Case 1. $d \notin b$. In this case, $d \in a$ and $d \notin b$, so $d \in a-b$. By disjunction introduction, $(d \in a-b \vee d \in a-c)$. So, $d \in (a-b) \cup (a-c)$.

Case 2. $d \notin c$. Similar to case 1.

Thus, $d \in (a-b) \cup (a-c)$.

($\leftarrow$) Assume $d \in (a-b) \cup (a-c)$. So, either $d \in (a-b)$ or $d \in (a-c)$.

Case 1. $d \in (a-b)$. Thus, $d \in a$ and $d \notin b$. Trivially, since $d \notin b$, $\neg(d \in b \ \& \ d \in c)$. So, $d \notin (b \cap c)$. Since $d \in a$ and $d \notin (b \cap c)$, we have that $d \in a-(b \cap c)$.

Case 2. $d \in (a-c)$. Similar to case 1.

Thus, $d \in a-(b \cap c)$.

Since d was arbitrary, we have shown that $\forall x(x \in (a-(b \cap c)) \leftrightarrow x \in (a-b) \cup (a-c))$. So, by extensionality (since both are obviously sets), we have **$(a - (b \cap c)) = (a - b) \cup (a - c)$**

B. Assume that **a** is a set. Prove that if **$\bigcup a \subseteq a$**, then **$a \subseteq \wp(a)$.**

Assume that **$\bigcup a \subseteq a$**. We know that a is a set and that $\wp(a)$ is a set, so it suffices to prove that every member of a is also a member of $\wp(a)$. Let b be an arbitrary member of a, i.e., assume $b \in a$.

We need to show that $b \in \wp(a)$. For this, it suffices to show that $b \subseteq a$. Let c be an arbitrary member of b. All we need to do is show that c also belongs to a.

We have that $c \in b$ and $b \in a$. So, $\exists x(c \in x \ \& \ x \in a)$, and thus $c \in \cup a$. We have assumed that $\cup a \subseteq a$, so $c \in a$. Since c was an arbitrary member of b, $b \subseteq a$. So, $b \in \wp(a)$.

Since b was an arbitrary member of a, it follows that $a \subseteq \wp(a)$.

C. Prove: If **R** and **S** are both symmetric on **A**, then **(R - S)** is symmetric on **A**.

Assume that R and S are both symmetric on A.
To show that (R-S) is also symmetric on A, we need to consider two arbitrary members of A, b and c. Assume that $\langle b,c\rangle \in (R-S)$. It suffices to show that $\langle c,b\rangle \in (R-S)$.

Since $\langle b,c\rangle \in (R-S)$, we know that $\langle b,c\rangle \in R$ and $\langle b.c\rangle \notin S$. Since R is symmetric on A, and both b and c belong to A, we have that $\langle c,b\rangle \in R$. Similarly, since S is symmetric on A, and we have that $\neg Sbc$, we also know that $\neg Scb$ (since, if Scb were true, then Sbc would also be true, by the symmetry of S on A). Thus, we have both $\langle c,b\rangle \in R$ and $\langle c,b\rangle \notin S$. So, $\langle c,b\rangle \in (R-S)$.
Since b and c were arbitrary members of A, we have shown that $\forall x \forall y((x \in A \ \& \ y \in A) \rightarrow ((R-S)xy \rightarrow (R-S)yx))$, i.e., that (R-S) is symmetric on A.