CHAPTER FOUR: QUANTIFIERS

Up to this point, we've used only sentence letters, which stand for simple complete sentences.

Now, we're going to break inside simple sentences, break them into subjects and predicates.

In our language, subjects = names, noun phrases

predicates = verb phrases

Verb phrases: -----is a man.

----is mortal.

----sleeps.

----golfs.

Combine with proper names & result in complete sentences, true or false.

Socrates is a man.

Socrates is mortal.
Reagan sleeps.
Quayle golfs.

In the formal language QL, proper names will be represented by lower-case letters from the beginning of the alphabet: a, b, c,...o.

These are called <u>constants</u>

With or without numerical subscripts (so, we have infinitely many).

<u>Predicates</u> will be upper-case letters, with or without subscripts.
All predicates will have numerical superscripts.

F^1 -- combines with one constant: F^1a, F^1b, etc.

like: ----is a man.

F^2 -- combines with two constants:
F^2ab, F^2de

like: ---- loves ----.

F^3 -- combines with three constants:
F^3abc, F^3aba

like: ----gave---- to -----.

As a convention, we will drop the numerical superscript: just look at the number of constants after the predicate letter.

QUANTIFIERS AND VARIABLES

$\forall x$: every x

$\exists x$: some x, at least one x

Socrates is bald. Ba
Everything is bald.

Intermediate stage: x is bald. Bx.

For every x, x is bald. $\forall xBx$

Something is bald.

For some (at least one) x, x is bald. $\exists xBx$.

Categorical sentences.

All swans are white.
For all x, if x is a swan, then x is white.
$\forall x(Sx \rightarrow Wx)$

Some swans are white.
For some x, if x is a swan, then x is white.
$\exists x(Sx \,\&\, Wx)$

No swans are white.
Either:
$\neg\exists x(Sx \,\&\, Wx)$
$\forall x(Sx \rightarrow \neg Wx)$

Snow, white, bald: are called monadic (1-place) predicates.

Polyadic predicates: e.g., 2-place: loves, fears, moves.

In general,

all, each, every ⇒ ∀

some, a(n) ⇒ ∃

Translating English into Logical Symbols

The language of predicate logic introduces two new logical symbols: the universal quantifier (∀) and the existential quantifier (∃). These can be used to translate the words something and everything:

Something is coming.

∃xCx

Everything is coming.

∀xCx

Every sentence in English contains a main verb. These verbs come in several kinds:

Intransitive verbs: sleeps, walks, thirsts.

These take a subject only.

Transitive verbs: loves, moves, irritates. These take a subject and a direct object.

Ditransitive verbs: gives (as in A gives B C). These take a subject, a direct object and an indirect object.

Verbs shall be translated by means of predicate letters. We shall follow these letters by one, two or three underscores, indicating whether the verb is intransitive, transitive or ditransitive.

walks ⇒ W_

likes ⇒ L_ _

gives ⇒ G_ _ _

As I described earlier, **proper names** shall be symbolized by lower-case letters from a through o (the individual constants). We will precede these constants by a question mark, to indicate that the name must be combined with a predicate in order to produce a complete sentence. An underscore shall stand for a missing name or variable, and a shadowed question mark shall stand for a missing predicate.

'John' ⇒ [**?** j]

Predicates consisting of 'to be' plus an adjective or common noun are also translated by means of a single predicate letter, followed by an underscore.

'is white' ⇒ W_

Consider the sentence, 'John walks'. We symbolize the name John as [**?** j], and the predicate '...walks' as [W_]. When these two are combined, the constant 'j' replaces the underscore in the predicate, and the predicate repaces the question mark in the subject. The result is:

Wj

Proper names are only one kind of term. Another kind is a **noun phrase** like 'a boy' or 'all circles', in which a determiner ('a', 'all') is combined with a common noun ('boy', 'circle') or common noun phrase. A common noun phrase can be created by starting with a common noun and adding a relative clause (a clause headed by 'that', 'who', 'whom' or 'which'), and adjective or a prepositional phrase. The class of common noun phrases can be defined as follows:

1. If $\mathcal{A}$ is a common noun, then [$\mathcal{A}$] is a common noun phrase.
2. If $\mathcal{A}$ is a noun phrase, and $\mathcal{B}$ is a relative clause or a prepositional phrase, then [$\mathcal{A}$ $\mathcal{B}$] is a noun phrase.
3. If $\mathcal{A}$ is a common noun phrase, and $\mathcal{B}$ is an adjective, then [$\mathcal{B}$ $\mathcal{A}$] is a common noun phrase.

So, the following are c-noun phrases:

boy

boy in Austin

boy who lives in Austin

dark-haired boy

dark-haired boy who lives in Austin

boy who lives in a capital city

A relative clause results from replacing the subject, direct object or indirect object of a complete sentence by 'that', 'who', 'whom' or 'which' (where 'who' must always replace the subject and 'whom' the direct object).

A prepositional phrase is the result of combining a preposition (like 'in', 'on', 'through', etc.) with a noun phrase.

First, it is necessary to specify the translation of the **determiners** of English. The following table gives the translation of the most common determiners. When an English word or phrase is followed by a prime symbol, the resulting expression is a name for the symbolic translation of that word or phrase.

every' = any' = all' = [$\forall x(\mathbf{?}_i\, x \rightarrow \mathbf{?}_j\, x)$]
a' = an' = some' = [$\exists x(\mathbf{?}_i\, x \,\&\, \mathbf{?}_j\, x)$]
no' = none' = [$\neg\exists x(\mathbf{?}_i\, x \,\&\, \mathbf{?}_j\, x)$]
only' = [$\forall x(\mathbf{?}_j\, x \rightarrow \mathbf{?}_j\, x)$]
everybody' = anybody' = [$\forall x(Px \rightarrow \mathbf{?}_i\, x)$]
somebody' = [$\exists x(Px \,\&\, \mathbf{?}_i\, x)$]

The subscriped letters i and j indicate that two different predicates should be used to replace the question marks. You should introduce new indices as you proceed through the English sentence.

Note that the crucial difference between every and only concerns the order of these indices: when translating 'every, put the first available predicate into the antecedent of the conditional, while in translating 'only' you should put the first predicate in the consequent instead.

Although the variable x is used throughout this table, in fact one should introduce a new variable every time you translate a new determiner in an English sentence. So, if you've already used x, you should translate the next determiner using y instead.

Some swans are white.
[∃x($?_i$ x & $?_j$ x)] [S_] [W_]
[∃x(Sx & $?_j$ x)] [W_]
∃x(Sx & Wx)

All swans are white.
$[\forall x(\ ?_i\ x \rightarrow\ ?_j\ x)]$ [S_]
$[W\_][\forall x(Sx \rightarrow\ ?_j\ x)]$ [W_]
$\forall x(Sx \rightarrow Wx)$

No swans are white.
$[\neg\exists x(?_i\ x\ \&\ ?_j\ x)]$ [S_] [W_]
$[\neg\exists x(Sx \rightarrow\ ?_j\ x)]$ [W_]
$\neg\exists x(Sx\ \&\ Wx)$

Relative clauses consist of relative pronouns combined with predicates. The relative pronouns 'that', 'who', 'whom' and 'which' are all translated in the same way:

that' = who' = whom' = which' =

$[(?_i\ \_\_i\ \&\ ?_j\ \_\_i\ )]$

In this translation, the question marks stand for two different predicates, but the underscores, which share the same index, stand for one individual.

Adjectives attach to common noun phrases and produce new common noun phrases. For example, the adjective 'white' can be translated as follows:

$$\text{white}' = [(W\_{}_{i} \ \& \ \mathbf{?}_{\mathbf{i}} \ \_{}_{i})]$$

The two underscores are linked by a common index (i) and must both be replaced by the same term. The question mark should be combined with the common noun or noun phrase to which the adjective is attached. Thus, the phrase 'a white swan' would be translated as follows:

$$[\exists x(\mathbf{?}_{\mathbf{i}}\ x \ \& \ \mathbf{?}_{\mathbf{j}}\ x)]\ [(W\_{}_{k} \ \& \ \mathbf{?}_{\mathbf{k}}\_{}_{k})\ ]$$

$$[S\_]$$

$$[\exists x(\mathbf{?}_{\mathbf{i}}\ x \ \& \ \mathbf{?}_{\mathbf{j}}\ x)]\ [(W\_{}_{k} \ \& \ S\_{}_{k})]$$

$$[\exists x((Wx \ \& \ Sx) \ \& \ \mathbf{?}_{\mathbf{j}}\ x)]$$

At this point, we are ready to combine this noun phrase with a predicate (replacing the remaining shadowed question mark), producing a complete formula.

Prepositional phrases consist of prepositions plus noun phrases. A preposition is translated by means of a two-place predicate, a predicate letter followed by two plain question marks. This predicate represents the spatial or temporal relationship encoded by the preposition. For example, the translation of the predicate 'in' includes the two-place predicate I_{i}_{j}, which represents the relationship of the thing corresponding to $\_{i}$'s being inside the thing represented by $\_{j}$. The full translation of a preposition is as follows:

$$in' = [(?_{\mathbf{i}}\ \_{i}\ \&\ I\_{i}\ \_{j}\ )]$$

The following translations illustrate the use of relative clauses and prepositional phrases:

All students who attend will do well.

$[\forall x(?_{i}\, x \rightarrow ?_{j}\, x)]\ [S\_]\ [\ [(?_{k}\ \_{}_{i}\ \&\ ?_{m}\ \_{}_{i}\ )]\ [A\_]\ [W\_]$

$[\forall x(?_{i}\, x \rightarrow ?_{j}\, x)]\ [(S\ \_{}_{i}\ \&\ ?_{m}\ \_{}_{i}\ )]\ [A\_]\ [W\_]$

$[\forall x(?_{i}\, x \rightarrow ?_{j}\, x)]\ [(S\ \_{}_{i}\ \&\ A\ \_{}_{i})]\ [W\_]$

$[\forall x((Sx\ \&\ Ax) \rightarrow\ ?_{j}\, x)]\ [W\_]$

$\forall x((Sx\ \&\ Ax) \rightarrow Wx)$

Some students who attend will do well.

$[\exists x(?_{i}\, x\ \&\ ?_{j}\, x\ )]\ [S\_]\ [\ [(?_{k}\ \_{}_{i}\ \&\ ?_{m}\ \_{}_{i}\ )]\ [A\_]\ [W\_]$

$[\exists x(?_{i}\, x\ \&\ ?_{j}\, x)]\ [(S\ \_{}_{i}\ \&\ ?_{m}\ \_{}_{i}\ )]\ [A\_]\ [W\_]$

$[\exists x(?_{i}\, x\ \&\ ?_{j}\, x)]\ [(S\ \_{}_{i}\ \&\ A\ \_{}_{i})]\ [W\_]$

$[\exists x((Sx\ \&\ Ax)\ \&\ ?_{j}\, x)]\ [W\_]$

$\exists x((Sx\ \&\ Ax)\ \&\ Wx)$

A boy in Austin likes Jane.

$[\exists x(?_i\, x \,\&\, ?_j\, x)]\ [B\_]\ [(?_m\, \_{}_i \,\&\, I\_{}_i\, \_{}_j)]\ [?_n\, a]\ [L\_{}_k\, \_{}_m]\ [?\, j]$

$[\exists x(?_i\, x \,\&\, ?_j\, x)]\ [B\_]\ [(?_m\, \_{}_i \,\&\, I\_{}_i\, a)][L\_{}_k\, j]$

$[\exists x(?_i\, x \,\&\, ?_j\, x)]\ [(B\_{}_i \,\&\, I\_{}_i\, a)][L\_{}_k\, j]$

$[\exists x((Bx \,\&\, Ixa) \,\&\, ?_j\, x)]\ [L\_{}_k\, j]$

$\exists x((Bx \,\&\, Ixa) \,\&\, Lxj)$

Every boy kisses a girl.

$[\forall x(?_i\, x \rightarrow ?_j\, x)]\ [B\_]\ [K\_{}_j\, \_{}_k]$
$\quad [\exists y(?_k\, y \,\&\, ?_m\, y)]\ [G\_{}_m]$

$[\forall x(?_i\, x \rightarrow ?_j\, x)]\ [B\_]\ [K\_{}_j\, \_{}_k]$
$\quad [\exists y(Gy \,\&\, ?_m\, y)]$

$[\forall x(?_i\, x \rightarrow ?_j\, x)]\ [B\_]\ [\exists y(Gy \,\&\, K\_{}_j\, y)]$

$[\forall x(Bx \rightarrow ?_j\, x)]\ [\exists y(Gy \,\&\, K\_{}_j\, y)]$

$\forall x(Bx \rightarrow \exists y(Gy \,\&\, Kxy))$

Pronouns (he, she, it, him, her) should be translated by means of constants or variables. In translating a pronoun in an English sentence, one should try to discern which preceding noun phrase in the sentence is the antecedent of the pronoun. Once you have identified this, you can use the corresponding constant or variable as your translation of the pronoun. For example:

Jane came, and Mike greeted her.

$$(Cj \mathbin{\&} Gmj)$$

A visitor arrived, and he asked for Sally.

$$\exists x((Vx \mathbin{\&} Ax) \mathbin{\&} A'xs)$$

Every student joined a club that accepted him.

$$\forall x(Sx \rightarrow \exists y(Cy \mathbin{\&} Ayx))$$

1. **Generic "a", "an":**
A friendship is forever.
2. **"A", "an", and "some" in antecedents of conditionals:**
If you love someone, you are fortunate.
If you love someone, you should be good to him/her.
3. **"Any" takes wider scope than "every".**
If you love anyone, you are fortunate.
If you love everyone, you are fortunate.
I didn't catch any fish.
I didn't catch a fish.
I didn't catch every fish.

4. **Only A's are B's = All B's are A's.**
Only mammals nurse their young.
Only Republicans are happy.

Vocabulary

Sentence letter constants: p, q, r,...
n-ary predicates: A, B,...Z.
Individual constants: a, b, c,...o.
Individual variables: t, u, v, w, x, y, z.
Sentential connectives: ¬, →, &, ∨, ↔.
Quantifiers: ∀, ∃.
Grouping indicators: (,).

Formation rules

1.Any sentence letter constant is a formula.
2. Any n-ary predicate followed by n constants is a formula.
3. If $\mathscr{A}$ is a formula, so is $\neg\mathscr{A}$.
4. If $\mathscr{A}$ and $\mathscr{B}$ are formulas, so are $(\mathscr{A} \rightarrow \mathscr{B})$, $(\mathscr{A}\ \&\ \mathscr{B})$, $(\mathscr{A} \vee \mathscr{B})$, and $(\mathscr{A} \leftrightarrow \mathscr{B})$.
5. If $\mathscr{A}$ is a formula with a constant c, and v is a variable that does not appear in $\mathscr{A}$, then $\exists v\mathscr{A}[v/c]$ and $\forall v\mathscr{A}[v/c]$ are formulas.
6. Every formula can be constructed by a finite number of applications of these rules.

Definition. An *interpretation*, M, of the language QL consists of a nonempty set D (M's *domain*, or *universe of discourse*) and a function φ assigning
(a) truth values to sentence letters,
(b) elements of D to constants, and
(c) sets of n-tuples of elements of D to n-ary predicates.

Symbol	Interpretation
Sentence letter	Truth-value (T,F)
Constant	Object in domain
n-ary predicate	Set of n-tuples in domain

1. P is true in M iff φ(P) = T.

2. $\mathcal{A}a_1 \ldots a_i$ is true in M iff
$<\varphi(a_1),\ldots\varphi(a_i)>$ belongs to φ($\mathcal{A}$).

3. φ(¬$\mathcal{A}$) = T iff φ($\mathcal{A}$) = F.
φ($\mathcal{A}$& $\mathcal{B}$) = T iff φ($\mathcal{A}$) = T and φ($\mathcal{B}$) = T.
φ($\mathcal{A}$∨$\mathcal{B}$) = T iff φ($\mathcal{A}$) = T or φ($\mathcal{B}$) = T.
φ($\mathcal{A}$→$\mathcal{B}$) = T iff φ($\mathcal{A}$) = F or φ($\mathcal{B}$) = T.
φ($\mathcal{A}$ ↔$\mathcal{B}$) = T iff φ($\mathcal{A}$) = φ($\mathcal{B}$).

4.
∃v$\mathcal{A}$ is true in M iff there is a constant c and a model M' which differs from M at most in assigning a different element of M's domain to c such that $\mathcal{A}$[c/v] is true in M'.

$\forall v \mathscr{A}$ is true in M iff for every constant c and every model M' which differs from M at most in assigning a different element of M's domain to c, $\mathscr{A}[c/v]$ is true in M'.

D = {Bob, Carol, Ted, Alice}
φ(F) = {Bob, Ted}
φ(G) = {Carol, Alice}
φ(R) = { <Bob, Carol>, <Carol, Bob>, <Ted, Alice>, <Bob, Bob>, <Alice, Bob>}

φ(S) = { <Carol, Bob, Ted>, <Bob, Carol, Alice>, <Ted, Alice, Carol>, <Alice, Bob, Alice> }

a,b, c: obvious
d: Ted.

Evaluate: Fa, Fb, Gc, Gb, Rbe, Rbb, Rba, Rbc, Raa

$\forall x(Fx \vee Gx)$ $\exists x(Fx \ \& \ Rxx)$

$\exists x \exists y((Fx \ \& \ Gy) \ \& \ (Rxy \ \& \ Ryx))$

$\forall x(Fx \rightarrow \exists yRyx)$
$\forall x \exists y \exists zSxyz$