Set rules:

Let t, t' stand for any closed term, y for any variable:

$$\frac{\neg St}{\forall y\; y \notin t} \qquad \frac{t \in t'}{St'} \qquad\qquad\qquad S$$

7. 1 EXTENSIONALITY

$$\forall x \forall y((Sx \,\&\, Sy) \rightarrow (\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y))$$

Derivable rule.

Let t and t' stand for any closed term, v for any variable:

$$\begin{array}{l} St \\ St' \\ \underline{\forall v(v \in t \leftrightarrow v \in t')\qquad\quad} \end{array}$$

$t = t'$ Ext.

$\subseteq$ Exploitation

$$\frac{t \subseteq t'}{St \ \&\ St' \ \&\ \forall v(v \in t \rightarrow v \in t')} \qquad \subseteq E$$

$$\frac{\neg St}{\neg t \subseteq t'} \qquad \subseteq E$$

$$\frac{\neg St'}{\neg t \subseteq t'} \qquad \subseteq E$$

New formation rule:

If v is a variable, t is a closed term, c is a constant, $\mathcal{A}$ is a formula containing c but not containing v, then:
$\{v \in t: \ \mathcal{A}[v/c]\}$ is a closed term.

Such a term is called an abstraction term. 'v' is the variable of abstraction, 't' is the restrictor term, and $\mathcal{A}$ [v/c] is the abstraction formula.

Abstraction axiom schema:
$\forall x\ (\ S\{z \in x : \mathcal{A}(z)\}\ \&$
$\forall y(\ y \in \{z \in x : \mathcal{A}(z)\} \quad \leftrightarrow (\ y \in x\ \&$
$\mathcal{A}(y))))$

Here $\mathcal{A}(z) = \mathcal{A}[z/c]$ for some $\mathcal{A}$ containing c and not z. Likewise, $\mathcal{A}(y) = \mathcal{A}[y/c]$.

Theorem 7.4: Any set x is the set of its self-identical members, $\{y \in x: y = y\}$

1. Show $\forall x(Sx \rightarrow (x = \{y \in x: y = y\})$
2. Show $Sa \rightarrow (a = \{y \in a: y = y\})$
3. Sa
4. $S\{y \in a: y = y\} \&$
   $\forall z(z \in \{y \in a: y=y\} \leftrightarrow$
   $(z \in a \ \& \ z=z))$ Abst.
5. $\forall z(z \in \{y \in a: y=y\} \leftrightarrow$
   $(z \in a \ \& \ z=z))$ &E, 4
6. Show $\forall z(z \in a \leftrightarrow z \in \{y \in a: y=y\})$
7. Show $c \in a \leftrightarrow c \in \{y \in a: y=y\}$
8. $c \in \{y \in a: y=y\} \leftrightarrow (c \in a \ \& \ c=c)$
   ∀E, 5
9. Show $c \in a \rightarrow c \in \{y \in a: y=y\}$
10. $c \in a$
11. $c=c$ =I
12. $c \in a \ \& \ c=c$ 10,11,&I
13. $c \in \{y \in a: y=y\}$ ↔E,8,12
14. Show $c \in \{y \in a: y=y\} \rightarrow c \in a$
15. $c \in \{y \in a: y=y\}$

16. $c \in a \ \& \ c = c$ ↔E, 8, 15
17. $c \in a$ 16, &E
18. $S\{ y \in a: y = y\}$ 4, &E
19. $a = \{ y \in a: y = y\}$ 3, 18, 6, Ext

Theorem 7.3: $\forall x \ S\{z \in x: A(z)\}$

Let v be any variable, t and t' any closed terms.
Let $\mathcal{A}(v) = \mathcal{A}[v/c]$, where $\mathcal{A}$ is some formula containing c and not v.
Let $\mathcal{A}(t) = \mathcal{A}[t/c]$.

Naive Abstraction:

$$\frac{t \in \{v \in t': A(v)\}}{t \in t' \ \& \ A(t)} \quad \text{Abstr.}$$

1.	Show $\forall x \exists y\ y \notin x$	
2.	Show $\exists y\ y \notin a$	
3.	~~Show~~ $\{z \in a: z \notin z\} \notin a$	
4.	$\{z \in a: z \notin z\} \in a$	
5.	$\{z \in a: z \notin z\} = b$	$=$I, $\exists$I, $\exists$E
6.	~~Show~~ $b \in b$	
7.	$b \notin b$	
8.	$b \in a$	$=$E, 4, 5
9.	$b \in a\ \&\ b \notin b$	&I, 7, 8
10.	$b \in \{z \in a: z \notin z\}$	Abst., 9
11.	$b \in b$	$=$E, 4, 10
12.	$b \in \{z \in a: z \notin z\}$	$=$E, 5, 6
13.	$b \in a\ \&\ b \notin b$	Abst., 12
14.	$b \notin b$	&E, 13
15.	$\exists y\ y \notin a$	$\exists$I, 3

New formation rules:

If t and t' are closed terms, then $\{t, t'\}$ is a closed term.

If $t_1, \ldots, t_i$ are closed terms, then $\{t_1, \ldots, t_i\}$ is a closed term.

Unit Set

$$\frac{x \in \{y\}}{x = y} \qquad \text{Unit}$$

Pair Set

$$\frac{u \in \{y, z\}}{u = y \vee u = z} \qquad \text{Pair set}$$

Enumeration

$$\frac{x \in \{y_1, \ldots, y_i\}}{x = y_1 \vee x = y_2 \vee \ldots \vee x = y_i} \qquad \text{Enum}$$

Everything belongs to some set.

1. Show $\forall x \exists y(Sy \,\&\, x \in y)$
2. Show $\exists$ $y(Sy \,\&\, a \in y)$
3. S{a} Thm 6
4. $a \in \{a\}$ Unit E*
5. $Sa \,\&\, a \in \{a\}$
6. $\exists y(Sy \,\&\, a \in y)$ $\exists$I, 5

Theorem 7.10. There is a set without members. $\exists x\,(Sx \,\&\, \forall y\; y \notin x)$

1. Show $\exists x(Sx \,\&\, \forall y\,(y \notin x\,)$
2. $S\{x\!: x \neq x\}$ Th 7.3
3. ~~Show~~ $\forall y\; y \notin \{x\!: x \neq x\}$
4. ~~Show~~ $c \notin \{x\!: x \neq x\}$
5. $c \in \{x\!: x \neq x\}$ AIP
6. $c \neq c$ Abst, 5
7. $c = c$ =I
8. $\exists x\,(Sx \,\&\, \forall y\; y \notin x)$ &I,2, 3,∃I

Definition: $\varnothing = \{x\!: x \neq x\}$

Derivable inference rule:

$$\frac{\varnothing}{t \notin \varnothing} \qquad \varnothing$$

Theorem 7.11 $S\varnothing \,\&\, \forall y\; y \notin \varnothing$

Theorem 7.12. $\varnothing$ is the only empty set.
$\forall x((Sx \mathbin{\&} \forall y\, y \notin x) \to x = \varnothing)$

1.	Show $\forall x((Sx \mathbin{\&} \forall y\, y \notin x) \to x=\varnothing)$	
2.	Show $(Sa \mathbin{\&} \forall y\, y \notin a) \to a=\varnothing$	
3.	$Sa \mathbin{\&} \forall y\, y \notin a$	ACP
4.	~~Show~~ $\forall z(z \in a \leftrightarrow z \in \varnothing)$	
5.	~~Show~~ $b \in a \leftrightarrow b \in \varnothing$	
6.	~~Show~~ $b \in a \to b \in \varnothing$	
7.	$b \in a$	ACP
8.	$b \notin a$	$\forall$E3,&E 3
9.	$b \in \varnothing$	!, 7, 8
10.	~~Show~~ $b \in \varnothing \to b \in a$	
11.	$b \in \varnothing$	ACP
12.	$b \notin \varnothing$	$\varnothing$
13.	$b \in a$	!,11,12
14.	$b \in a \leftrightarrow b \in \varnothing$	$\leftrightarrow$I, 6,10
15.	$a = \varnothing$	Ext, 4,3,Thm 7.11

Corollary 7.1
$\forall x\, ((Sx \mathbin{\&} x \neq \varnothing) \to \exists y\, y \in x)$