1. Show $\forall x \forall y(x \in y \rightarrow y \neq \varnothing)$
2. Show $a \in b \rightarrow b \neq \varnothing$
3. $a \in b$
4. ~~Show~~ $b \neq \varnothing$
5. $b = \varnothing$
6. $a \notin b$ $\varnothing$
7. $a \in b$ R, 3

Unions

$\cup$

$$\frac{x \in y \cup z}{x \in y \vee x \in z} \qquad \cup$$

$\cup$#n

$$\frac{x \in y_1 \cup \ldots \cup y_i}{x \in y_1 \vee \ldots \vee x \in y_i} \qquad \cup\#n$$

1. Show $\forall x \forall y((Sx \& Sy) \rightarrow (x \subseteq y \rightarrow x \cup y = y))$
2. Show $(Sa \& Sb) \rightarrow (a \subseteq b \rightarrow a \cup b = b)$
3. $(Sa \& Sb)$
4. Show $a \subseteq b \rightarrow a \cup b = b$
5. $a \subseteq b$
6. ~~Show~~ $\forall z(z \in a \cup b \leftrightarrow z \in b)$
7. ~~Show c~~ $\in a \cup b \leftrightarrow c \in b$
8. ~~Show~~ $\rightarrow$
9. $c \in a \cup b$
10. $c \in a \vee c \in b$ — $\cup$, 9
11. Show $c \in a \rightarrow c \in b$
12. $c \in a$
13. $c \in b$ — $\subseteq$*, 5, 12
14. Show $c \in b \rightarrow c \in b$
15. $c \in b$
16. $c \in b$ — $\vee$E, 10,11, 14
17. ~~Show~~ $\leftarrow$
18. $c \in b$
19. $c \in a \vee c \in b$ — I, 18
20. $c \in a \cup b$ — $\cup$, 19
21. $a \cup b = b$ — Ext, 3, 4

Intersections

$\cap$

$$\frac{w \in x \quad \cap y}{w \in x \;\&\; w \in y} \qquad \cap$$

$\cap$ #n

$$\frac{x \in y_1 \cap \ldots \quad \cap \; y_n}{x \in y_1 \;\&\; \ldots \;\&\; x \in y_n} \qquad \cap\#$$

Defn -

$$\frac{x \in (y - z)}{x \in y \;\&\; x \notin z} \qquad \text{Defn -}$$

Definition. x and y are **disjoint** iff their intersection is empty. Or,

$\forall x \forall y$(x and y are disjoint $\leftrightarrow$ $x \cap y = \varnothing$).

djt E

$$\frac{\text{x and y are disjoint}}{x \cap y = \varnothing} \quad \text{djt E}$$

If $b \notin a$, then a and $\{b\}$ are disjoint.

1. Show $b \notin a \rightarrow a \cap \{b\} = \varnothing$
2. $b \notin a$
3. Show $\forall x(x \in a \cap \{b\} \leftrightarrow x \in \varnothing)$
4. ~~Show~~ $c \in a \cap \{b\} \leftrightarrow c \in \varnothing$
5. ~~Show~~ $\rightarrow$
6. $c \in a \cap \{b\}$
7. $c \in a \;\&\; c \in \{b\}$ — $\cap$,6
8. $c = b$ — &E, 7, Unit
9. $c \notin a$ — =E, 2, 8
10. $c \in a$ — &E, 7
11. $c \in \varnothing$ — !, 9, 10
12. ~~Show~~ $\leftarrow$
13. $c \in \varnothing$
14. $c \notin \varnothing$ — $\varnothing$
15. $c \in a \cap \{b\}$ — !, 13, 14
16. $a \cap \{b\} = \varnothing$ — Ext, 3

∪1

$$\frac{x \in \bigcup y}{\exists z(z \in y \,\&\, x \in z)} \qquad \bigcup 1$$

∩1

$$\frac{x \in \bigcap y}{\forall z(z \in y \rightarrow x \in z)} \qquad \bigcap 1$$

1. Show$(Sb \& \forall x(x \in a \rightarrow x \subseteq b)) \rightarrow \cup a \subseteq b$

2. $Sb \& \forall x(x \in a \rightarrow x \subseteq b)$

3. ~~Show~~ $\forall y(y \in \cup a \rightarrow y \in b)$

4. ~~Show~~ $c \in \cup a \rightarrow c \in b$

5. $c \in \cup a$

6. $\exists z(z \in a \& c \in z)$ — $\cup$E, 5

7. $d \in a \& c \in d$ — $\exists$E, 6

8. $d \in a \rightarrow d \subseteq b$ — $\forall$E, 2

9. $d \subseteq b$ — SL, 7, 8

10. $\forall w (w \in d \rightarrow w \in b)$ — $\subseteq$E, 9

11. $c \in d \rightarrow c \in b$ — $\forall$E, 10

12. $c \in b$ — SL, 7, 11

13. $\cup a \subseteq b$ — $\subseteq$E, Th 12, 2, 3

Exercise: show that $a \cap b$ is the same set as $\cap \{a,b\}$

1. Show $\forall x \forall y(x \cap y = \cap\{x,y\})$
2. Show $a \cap b = \cap\{a,b\}$
3. ~~Show~~ $\forall x(x \in a \cap b \leftrightarrow x \in \cap\{a,b\})$
4. ~~Show~~ $c \in a \cap b \leftrightarrow c \in \cap\{a,b\}$
5. ~~Show~~ $\rightarrow$
6. $c \in a \cap b$
7. $c \in a \ \& \ c \in b$
8. ~~Show~~ $\forall w(w \in \{a,b\} \rightarrow c \in w)$
9. ~~Show~~ $d \in \{a,b\} \rightarrow c \in d$
10. $d \in \{a,b\}$
11. $d=a \vee d=b$ Pair, 10
12. ~~Show~~ $d=a \rightarrow c \in d$
13. $d = a$
14. $c \in d$ &E, 7, =E, 13
15. ~~Show~~ $d = b \rightarrow c \in d$

    [Similar]
16. $c \in d$ $\vee$E, 11, 12, 15
17. $c \in \cap\{a,b\}$ $\cap$, 8

18. ~~Show~~ ←

19. $c \in \cap\{a,b\}$

20. $\forall w(w \in \{a,b\} \rightarrow c \in w)$ $\cap$, 19

21. $a \in \{a,b\}$ Th 8.1

22. $a \in \{a,b\} \rightarrow c \in a$ $\forall$E, 20

23. $c \in a$ $\rightarrow$E, 21,22

24. $c \in b$ [Similar]

25. $c \in a$ & $c \in b$ &I, 23, 24

26. $c \in a \cap b$ $\cap$, 25

27. $a \cap b = \cap\{a,b\}$

Ext, 3, Th 7.15, Th 7.22

℘ E

$$\frac{x \in \wp(y)}{x \subseteq y} \qquad \wp$$

Assume a is a set.

1. Show $\cup\wp(a) = a$
2. Show $\forall x(x \in \cup\wp(a) \leftrightarrow x \in a)$
3. Show $b \in \cup\wp(a) \leftrightarrow b \in a$
4. Show $\rightarrow$
5. $b \in \cup\wp(a)$
6. $\exists y(b \in y \ \&\ y \in \wp(a))$ — $\cup 1, 5$
7. $b \in c \ \&\ c \in \wp(a)$ — $\exists E, 6$
8. $c \subseteq a$ — $\&E, 7, \wp$
9. $b \in a$ — $\subseteq E^*, 7, 8$
10. Show $\leftarrow$
11. $b \in a$
12. $a \subseteq a$ — Th 1
13. $a \in \wp(a)$ — $\wp E, 12$
14. $\exists y(b \in y \ \&\ y \in \wp(a))$ — $\exists I, 11, 13$
15. $b \in \cup\wp(a)$ — $\cup 1, 14$
16. $\cup\wp(a) = a$ — Ext, Th. 7.18, Ass.

Assume a and b are sets.

1. Show $a \in b \rightarrow \wp(a) \in \wp(\wp(\cup b))$
2. $a \in b$
3. Show $\forall x(x \in \wp(a) \rightarrow x \in \wp(\cup b))$
4. Show $c \in \wp(a) \rightarrow c \in \wp(\cup b)$
5. $c \in \wp(a)$
6. $c \subseteq a$ $\wp$E, 5
7. ~~Show~~ $\forall y(y \in c \rightarrow y \in \cup b)$
8. ~~Show~~ $d \in c \rightarrow d \in \cup b$
9. $d \in c$
10. $d \in a$ $\subseteq$E*, 6, 9
11. $\exists z(d \in z \ \&\ z \in b)$ $\exists$I. 2, 10
12. $d \in \cup b$ $\cup$E, 11
13. $c \subseteq \cup b$ $\subseteq$E, 6($\subseteq$E), Th 7.18, 7
14. $c \in \wp(\cup b)$ $\wp$E, 13
15. $\wp(a) \subseteq \wp(\cup b)$ $\subseteq$E, Th 23, 3
16. $\wp(a) \in \wp(\wp(\cup b))$ $\wp$E, 15

Transitive sets

$\forall x($ x is transitive $\leftrightarrow$
$(Sx \,\&\, \forall y \forall z((y \in z \,\&\, z \in x) \rightarrow y \in x)))$

A set of sets is transitive iff all members of it are subsets of it.

1. $Sa \ \& \ \forall x(x \in a \rightarrow Sx)$ A

2. Show a is transitive $\leftrightarrow \forall y(y \in a \rightarrow y \subseteq a)$

3. ~~Show~~ $\rightarrow$

4. a is transitive

5. $Sa \ \& \ \forall y \forall z((y \in z \ \& \ z \in a) \rightarrow y \in a)$ Def Trans., 4

6. ~~Show~~ $\forall y(y \in a \rightarrow y \subseteq a)$

7. ~~Show~~ $b \in a \rightarrow b \subseteq a$

8. $b \in a$

9. ~~Show~~ $\forall z(z \in b \rightarrow z \in a)$

10. ~~Show~~ $c \in b \rightarrow c \in a$

11. $c \in b$

12. $c \in a$ $\forall E^2$, 5, etc.

13. $b \subseteq a$ $\subseteq$, 9

14. ~~Show~~ $\leftarrow$

15. $\forall y(y \in a \rightarrow y \subseteq a)$

16. ~~Show~~ $\forall y \forall z((y \in z \ \& \ z \in a) \rightarrow y \in a)$

17. ~~Show~~ $(d \in e \ \& \ e \in a) \rightarrow d \in a$

18. $d \in e \ \& \ e \in a$

19. $e \subseteq a$ $\forall E$, 15, 18

20. $d \in a$ $\subseteq$, 19, $\forall E$, 18

21. a is transitive Def Trans, 1, 16

Assume a and b are sets.

1. Show $\wp(a - b) \neq \wp(a) - \wp(b)$

2. $\wp(a - b) = \wp(a) - \wp(b)$

3. ~~Show~~ $\forall x(x \in \varnothing \rightarrow x \in a - b)$

4. ~~Show~~ $c \in \varnothing \rightarrow c \in a - b$

5. $c \in \varnothing$

6. $c \notin \varnothing$ $\varnothing$

7. $\varnothing \subseteq a - b$ $\subseteq$, 3, Th 7.11, Th 7.17

8. $\varnothing \in \wp(a-b)$ $\wp$, 7

9. ~~Show~~ $\forall x(x \in \varnothing \rightarrow x \in b)$

10. Similar

11. $\varnothing \subseteq b$ $\subseteq$, 9, Th 10, A

12. $\varnothing \in \wp(b)$ $\wp$, 11

13. Show $\varnothing \notin \wp(a) - \wp(b)$

14. $\varnothing \in \wp(a) - \wp(b)$

15. $\varnothing \notin \wp(b)$ Def -, 14, &E

16. $\varnothing \in \wp(b)$ R, 12

17. $\varnothing \in \wp(a) - \wp(b)$ =E, 2, 8

1.	Show $\forall y(y \in b \rightarrow y \cup a \in \wp(a \cup \cup b))$	
2.	Show $c \in b \rightarrow c \cup a \in \wp(a \cup \cup b)$	
3.	$c \in b$	
4.	Show $\forall z(z \in c \cup a \rightarrow z \in a \cup \cup b)$	
5.	Show $e \in c \cup a \rightarrow e \in a \cup \cup b$	
6.	$e \in c \cup a$	
7.	$e \in c \vee e \in a$	$\cup$, 6
8.	~~Show~~ $e \in c \rightarrow e \in a \cup \cup b$	
9.	$e \in c$	
10.	$\exists y(e \in y \,\&\, y \in b)$	$\exists$I, 9, 3
11.	$e \in \cup b$	$\cup$, 10
12.	$e \in a \vee e \in \cup b$	$\vee$I, 11
13.	$e \in a \cup \cup b$	$\cup$, 12
14.	~~Show~~ $e \in a \rightarrow e \in a \cup \cup b$	
15.	$e \in a$	
16.	$e \in a \cup \cup b$	$\vee$I, 15, $\cup$
17.	$e \in a \cup \cup b$	$\vee$E, 7, 8, 14
18.	$c \cup a \subseteq a \cup \cup b$	$\subseteq$, 4, Th7.14
19.	$c \cup a \in \wp(a \cup \cup b)$	$\wp$, 18

Definitions (using naive abstraction)

Unit Set Definition:

$$\{a\} = \{ \quad x: x = a\}$$

Pair set, Enumeration set definitions:

$\{a,b\} = \{x: x = a \vee x = b\}$

$\{a_1, a_2, ..., a_n\} =$

$$\{x: x = a_1 \vee x = a_2 \vee .... x = \qquad a_n\}$$

Empty set:

$$\varnothing = \{x: x \neq x\}$$

Pairwise union:

$$a \cup b = \{x: x \in a \vee x \in b\}$$

Pairwise intersection:

$$a \cap b = \{x: x \in a \ \& \ x \in b\}$$

Relative complement:

$$a - b = \{x: x \in a \ \& \ x \notin b\}$$

Union of a single set:

$$\cup a = \{x: \exists y(x \in y \ \& \ y \in a)\}$$

Intersection of a single set:

$$\cap a = \{x: \forall y(y \in a \rightarrow x \in y)\}$$

Power set:

$$\wp(a) = \{x: x \subseteq a\}$$