1. Show a∪a = a
2. Sa∪a Th 1.5.1
3. Sa A
4. Show ∀x(x ∈ a∪a ↔ x ∈ a)
5. Show b ∈ a∪a ↔ b ∈ a
6. Show →
7. b ∈ a ∪a
8. b ∈ a ∨ b ∈ a ∪, 7
9. Show b ∈ a
10. b ∉ a
11. b ∈ a ∨E*, 8, 10
12. Show ←
13. b ∈ a
14. b ∈ a ∨ b ∈ a ∨I, 13
15. b ∈ a∪a ∪, 14
16. a∪a = a Ext, 2, 3, 4

#4.

1. Show $a \cup (b \cup c) = (a \cup b) \cup c$
2. $S a \cup (b \cup c)$ Th 1.5.1
3. $S (a \cup b) \cup c$ Th 1.5.1
4. Show $\forall x(x \in a \cup (b \cup c) \leftrightarrow x \in (a \cup b) \cup c)$
5. Show $d \in a \cup (b \cup c) \leftrightarrow d \in (a \cup b) \cup c$
6. Show $\rightarrow$
7. $d \in a \cup (b \cup c)$
8. $d \in a \vee d \in b \cup c$ $\cup$, 7
9. $d \in a \vee (d \in b \vee d \in c)$
   $\cup$, 8 (replacement)
10. $(d \in a \vee d \in b) \vee d \in c$
    Assoc. of $\vee$, 9
11. $d \in a \cup b \vee d \in c$ $\cup$, 10
12. $d \in (a \cup b) \cup c$ $\cup$, 11
13. Show $\leftarrow$
    Similar
14. $a \cup (b \cup c) = (a \cup b) \cup c$ Ext, 2, 3, 4

#12.

1. Show a - b $\subseteq$ a
2. Show $\forall$x(x $\in$ a - b $\rightarrow$ x $\in$ a)
3. Show c $\in$ a - b $\rightarrow$ c $\in$ a
4. c $\in$ a - b
5. c $\in$ a & c $\notin$ b Def -, 4
6. c $\in$ a
7. a - b $\subseteq$ a $\subseteq$, 2, Th 1.5.5, A

14.

Show a $\cap$b = b $\cap$a. Both are sets by Th 1.5.3, so we need $\forall$x(x $\in$ a $\cap$ b $\leftrightarrow$ x $\in$ b $\cap$ a).

($\rightarrow$). Assume c $\in$ a $\cap$ b. Then c $\in$ a & c $\in$ b, by $\cap$. By commutativity of &, c $\in$ b & c $\in$ a, so c $\in$ b $\cap$ a, by $\cap$.

($\leftarrow$) Similar.

So, a$\cap$b = b$\cap$a, by Extensionality.

16. Assume a $\subseteq$ c and b $\subseteq$ c. To show that a - b = a $\cap$(c-b), since both are sets, we can use Extensionality.
($\rightarrow$) d $\in$ a - b. So, d $\in$ a & d $\notin$ b, by Def. -. Since a $\subseteq$ c, d $\in$ c. Since d $\in$ c & d $\notin$ b, d $\in$ c-b, by Def. -. Since d $\in$ a & d $\in$ c - b, d $\in$ a $\cap$ (c-b), by $\cap$.
($\leftarrow$) d $\in$ a $\cap$ (c - b). By $\cap$ and Def. -, we get: d $\in$ a & d $\in$ c & d $\notin$ b. So, d $\in$ a & d $\notin$ b, hence d $\in$ a - b.

22. Suppose that a $\subseteq$ b. To show that a ∩ c $\subseteq$ b ∩ c, since both are sets by Th. 1.5.3, it is enough to show that if d ∈ a ∩c, then d ∈ b ∩ c. Suppose d ∈ a ∩ c. Then d ∈ a & d ∈ c. Since a $\subseteq$ b, d ∈ b. So d ∈ b & d ∈ c. Hence, d ∈ b ∩ c, by ∩. Thus, by $\subseteq$, a ∩ c $\subseteq$ b ∩ c.

10. Suppose ∀x(x ∈ a → x∩b = ∅). Show that ∪a ∩ b = ∅. Suppose for contradiction that c ∈ ∪a ∩ b. Then, by ∩, c ∈ ∪a & c ∈ b. Since c ∈ ∪a, there is a d such that c ∈ d & d ∈ a (by ∪1). By ∀E, since d ∈ a, d∩b = ∅. Since c ∈ d & c ∈ b, c ∈ d∩b, by ∩. By =E, c ∈ ∅. But c ∉ ∅, a contradiction. Therefore, ∪a ∩ b has no members. Since it is a set, by Th 1.4.3, it is identical to the empty set. So, ∪a and b are disjoint.

6. Assume $\forall x(x \in a \rightarrow Sx)$. Show $a \subseteq \wp(\cup a)$. Assume $b \in a$. By hypothesis, b is a set. Show: $b \subseteq \cup a$. Assume $c \in b$. Since $b \in a$, $c \in \cup a$. So, $b \subseteq \cup a$, and $b \in \wp(\cup a)$.

8. Show $\wp(a) \cup \wp(b) \subseteq \wp(a \cup b)$. Assume $c \in \wp(a) \cup \wp(b)$. So, either $c \in \wp(a)$ or $c \in \wp(b)$. Case 1. $c \in \wp(a)$. So, $c \subseteq a$. Therefore, $c \subseteq a \cup b$, and $c \in \wp(a \cup b)$. Case 2 is symmetrical.

10. Show $a \subseteq b$ iff $\wp(a) \subseteq \wp(b)$.
(→) Assume $a \subseteq b$. Show $\wp(a) \subseteq \wp(b)$. Assume $c \in \wp(a)$. So, $c \subseteq a$. Since $a \subseteq b$, $c \subseteq b$. So, $c \in \wp(b)$.
(←) Assume $\wp(a) \subseteq \wp(b)$. Show $a \subseteq b$. Assume $c \in a$. Then, $\{c\} \subseteq a$, and $\{c\} \in \wp(a)$. Since $\wp(a) \subseteq \wp(b)$, $\{c\} \in \wp(b)$, and $\{c\} \subseteq b$. Therefore, $c \in b$.

Complex abstraction terms -- formation rule

If t* is an open term which contains free variables $v_1,...,v_i$, and and φ is an open formula with at most $v_1,\ldots, v_i$ free, then:

{t* [$v_1,...,v_i$]: φ [$v_1,...,v_i$] }
is a closed term.

Examples:

$$\{\mathbb{D}(x):\ x \in a\}$$

$$\{<x,y>:\ x \in a\ \&\ y \in b\}$$

$$\{\cap(x):\ x \in a\ \&\ c \notin x\}$$

$$\{<x,y>:\ x \in \cap(R)\ \&\ y \in \mathbb{D}(R)\ \&\ Ryx\}$$

$$\{<x,y>:\ x \in A\ \&\ y \in \cap(R)\ \&\ Rxy\}$$

$$\{\wp(x):\ x \in a\}$$

$$\{x \cup y:\ x,\ y \in a\ \&\ x \neq y\}$$

C-Abstraction*.

$$\frac{A(c_1 \ldots c_i)}{t^*(c_1,\ldots,c_i) \in \{t^*(v_1 \ldots v_i) : A(v_1 \ldots v_i)\}}$$

(Note: C-Abstr* is **not** reversible)

C-Abstr

$c \in \{t^*(v_1\ldots\ v_i) : v_i \in\ \ t_i\ \&\ A(v_1\ldots\ v_i)\ \}$

$\exists x_1\ldots\exists x_i(\ c = t^*(x_1\ldots x_i)\ \ A(x_1\ldots x_i)\ )$

Here are some examples of using C-Abstr:

a ∈ R & b ∈ S & a ∉ b
∴ <a,b> ∈ {<x,y>: x∈R & y∈S & x∉y}

a ∈ b
∴ Ɗ(a) ∈ {Ɗ(x): x ∈ b}

a ∈ b & c ∉ a
∴ ∩(a) ∈ { ∩(x) : x ∈ b & c ∉ x)

b ∈ a
∴ b^{-1} ∈ {x^{-1}: x ∈ a}

Finally, some examples of C-Abstr in the top-down direction:

$a \in \{<x,y>: x \in R \ \& \ y \in R\}$
$\therefore \exists x \exists y(a = <x,y> \ \& \ x \in R \ \& \ y \in R)$

$c \in \{\bigcap(y): y \in b \ \& \ a \cap y = \varnothing\}$
$\therefore \exists y(c = \bigcap(y) \ \& \ y \in b \ \& \ a \cap y = \varnothing)$