Strong Induction on N, for n ≥ m

Show $\forall x(x \geq m \rightarrow A)$
Show $A[m/x]$
.
.
Assume $k > m$ & $\forall y(m \leq y < k \rightarrow A[y/x])$
Show $A[k/x]$
.
.
$A[k/x]$

Example:
Show for every $x \geq 2$, there is a prime y and a number $z > 0$ such that $x = y \cdot z$.

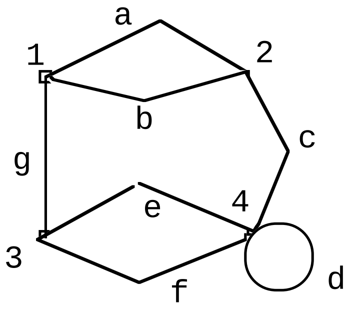
If $n = 2$, $2 = 2 \cdot 1$, 2 is a prime and $1 > 0$.

Suppose for every x such that $2 \leq x < n$, there is a prime y and number $z > 0$ such that $x = y \cdot z$.
Show: there is a prime y and a number z such that $n = y \cdot z$.

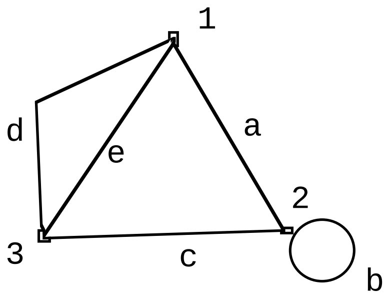
Either n is prime or not.
Case 1. If n is prime, then $n = n \cdot 1$, n is prime, and $1 > 0$. Let n be y and 1 be z.
Case 2. If n is not prime, then for some a and b greater than 1, $n = a \cdot b$, and a and b are less than n (show this by induction).

By ind hyp, $a = m \cdot q$, for some prime m and $q > 0$. By assoc. of mult., $n = m \cdot (q \cdot b)$. Since $q > 0$ and $b > 1$, $(q \cdot b) > 1$. Let m be y and let $(q \cdot b)$ be z.

A graph is a triple <A,B,f>, where A is the set of edges, B is the set of vertices, and f is a function from A into the set of pair sets of objects from B, that is, $\{\{x,y\}: x \in B \ \&\ y \in B\}$. The function f assigns two endpoints from B to each edge in A.



A



B

The *degree* of a vertex in a graph is the number of ends of edges which meet at that vertex.

Definition. A path p, {< $a_1,..,a_n$>, < $b_1,...,b$n+1>}, is **closed** iff $b_1 = b$ n+1.

Definition. A path is **simple** iff all of its edges are distinct.

Definition. A path p, {< $a_1,..,a_n$>, < $b_1,...,b$n+1>}, is a **cycle** iff p is closed and simple and the vertices b_2 through bn+1 are distinct.

Definition. A graph is **connected** iff for every vertex in the graph, there is a path in the graph from one to the other.

An Euler circuit in graph C is a closed path in C which uses each edge of C exactly once.

Theorem 10.18 . A finite connected graph G in which every vertex has an even degree has an Euler circuit.

Lemma 10.3 If G is a finite connected graph in which every vertex has an even degree, then there is at least one cycle in G.
(Lemma is proved on p. 124)

[Proof of the theorem] By strong induction on the number of edges.

(i) Base case. Suppose G has only one edge. Then, it must be a loop, else b_1 would not have an even degree. This loop is an Euler circuit for G.

(ii) Inductive case. Suppose G has n edges, $n > 1$.

Inductive hypothesis: Every finite connected graph with fewer than n edges in which every vertex has an even degree has an Euler circuit.

By Lemma 10.3, we know that G has at least one cycle C. Let G\C be the subgraph which is obtained by removing from G all the edges of C and all the vertices on C of degree 2. Removing the edges of C removes an even number from the degree of each vertex of G.

Consequently, all the vertices in G\C have even degree in G\C.
G\C consists of a finite number of connected subgraphs, $H_1,...,H_k$. Each of these connected subgraphs H_i has fewer than n edges, so, by the inductive hypothesis, each of them has an Euler circuit, p_i. Since G was connected, each of these subgraphs has at least one vertex on the cycle C. For each H_i, choose such a vertex h_i on C.

Therefore, G has an Euler circuit, which can be constructed in the following manner.

(i) Start with the initial vertex b_1 on C.

(ii) Proceed along C until you come to the first vertex chosen as one of the h_i's. At this point, leave the cycle C and follow instead the Euler circuit p_i.

(iii) Upon returning to h_i, proceed again along C until the next chosen vertex is reached.

Eventually, all of the edges will be traversed exactly once, and one will return again to b_1.