# Practice Exam 4

I. Multiple Choice.

1. Which of the following binary relations (on N) is **not** a function from N into N? [Think of x as the input and y as the output.]

(a) $Rxy \leftrightarrow y = x + 3$

(b) $Sxy \leftrightarrow y = x - 1$

(c) $Txy \leftrightarrow y = x$

(d) $Uxy \leftrightarrow y = x^2 + 5$

(e) $Vxy \leftrightarrow y = 2^x$

2. If $f:A \to B$ is bijective, then which is **not** necessarily true of $f^{-1}$?

(a) $f^{-1}:B \to A$ is bijective.

(b) $\forall x \in A\ f^{-1}(f(x)) = x$.

(c) $\forall x \in B\ f(f^{-1}(x)) = x$.

(d) $f = f^{-1}$.

(e) $f = (f^{-1})^{-1}$.

3. Which of the following functions is surjective (onto)?

(a) $f:N \to N$, where $f(x) = x + 2$.

(b) $g:N \to N$, where $g(x) = 3 \cdot x - 1$.

(c) $h:Z \to Z$, where $h(x) = -x$.

(d) $j:Z \to Z$, where $j(x) = x^2$.

(e) $k:R \to R$, where $k(x) = x^2$.

4. Which of the following is **not** true of N (the set of natural numbers)?
(a) N is inductive.
(b) N is infinite.
(c) $<$ on N is well-founded.
(d) Every nonempty subset of N has a least member.
(e) Every member of N is the successor of some member of N.

5. What is wrong with the following would-be proof?
[Assume the following definition: y is odd $\leftrightarrow \exists z(z \in N \;\&\; y = 2 \cdot z + 1)$.]

"Show: for all natural numbers x, if $x \geq 1$, then $x^2 + x$ is odd.
By induction on x, starting with x = 1.
Base case: Show: 1 is odd.
$1 = 2 \cdot 0 + 1$, so 1 is odd, by definition.

Inductive hypothesis: Assume $n^2 + n$ is odd. That is, for some m, $n^2 + n = 2 \cdot m + 1$
Show: $(n+1)^2 + (n+1)$ is odd.

$$
\begin{aligned}
(n+1)^2 + (n+1) &= (n^2 + 2n + 1) + (n + 1) \\
&= (n^2 + n) + (2n + 2) \\
&= (n^2 + n) + 2 \cdot (n + 1)
\end{aligned}
$$

By inductive hypothesis, $(n^2 + n) = 2 \cdot m + 1$, for some m. Therefore:

$$
\begin{aligned}
(n + 1)^2 + (n + 1) &= (2 \cdot m + 1) + 2 \cdot (n + 1) \\
&= 2 \cdot (m + n + 1) + 1
\end{aligned}
$$

Therefore, $(n+1)^2 + (n+1)$ is odd, by definition."

(a) Error in the statement of the base case.
(b) Error in the proof of the base case.
(b) Error in the statement of the inductive hypothesis.
(c) Error in the statement of the inductive case.
(d) Error in the proof of the inductive case.

6. What's wrong with the following would-be proof?
"Show: for all numbers x, y and z, $(x + y)\cdot z = x\cdot z + y\cdot z$.
By induction on x, starting with x = 0.
Base case: Show: for all numbers x and y, $(0 + y)\cdot z = 0\cdot z + y\cdot z$.
By Cor. 38.1, $0 + y = y$, so $(0 + y) \cdot z = y \cdot z$. By Cor 38.1 again, $y\cdot z = 0 + y\cdot z$. By Cor. 39.1, $0\cdot z = 0$. So, $0 + y \cdot z = 0 \cdot z + y \cdot z$.
Inductive hypothesis: for all numbers y and z, $(n+y)\cdot z = n\cdot y + y\cdot z$.
Show: for all numbers y and z, $((n+1) + y)\cdot z = (n+1)\cdot y + y\cdot z$.

$((n+1) + y)\cdot z = ((n+y) + 1)\cdot z$, by Cor 38.2,
$= (n+y)\cdot z + z$, by Cor. 39.2.
$= (n\cdot z + y\cdot z) + z$, by IH.
$= (n\cdot z + z) + y\cdot z$, by commutativity and associativity of +.
$= (n+1)\cdot z + y\cdot z$, by Cor. 39.2.

Answers:
(a) Error in Base case.
(b) Error in statement of inductive hypothesis.
(c) Error in statement of the Show line for inductive step.
(d) Error in proof of inductive step.
(e) No serious errors.

7. Consider the following definition of $2^{*n}$, "2 to the super-power of n":

$2^{*0} = 2$
$2^{*(k+1)} = 2^{2^{*k}}$

Which of the following correctly lists the first four values of this function (i.e., $2^{*0}, 2^{*1}, 2^{*2}, 2^{*3}$)?
(a) 2, 4, 16, 256
(b) 2, 4, 16, 128
(c) 2, 4, 16, 64
(d) 2, 4, 16, 65536
(e) 0, 2, 4, 16

II. Proofs by Induction. Use your blue books.

8. $$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

9. Consider the formal language consisting of vocabulary items 📖 and ⌛. Grammatical strings are defined recursively as follows:

(1) List(📖) is a string.
(2) If 𝒜 is a string, then List(📖)_𝒜_List(⌛) is a string.
(3) There are no other strings.

Prove that in any string, the number of 📖's is greater than the number of ⌛'s.

## Bonus Problem. (10 points maximum)

Use the following recursive definition of the concatenation function on lists :

1. If x is a list, then x^NIL = x.

2. If x and y are lists, then x^Cons(y,a) = Cons(x^y, a).

Prove that for all lists x, y and z, (x^y)^z = x^(y^z). [Hint: prove by induction on the length of z.]