First-order Predicate Logic

Odysseusis hurting me.

Nobody is hurting me.

Predicates:

_______ is hurting me.

_______is hurting _________.

____ gives _____ to _______.

Predicates represented by upper¬case letters.
If followed by no names: complete statement.
If followed by one name: quality or property.
If followed by two names: binary relation.
etc.

Nonvariable (constant) name: an expression used to name a particular, individual thing (person, place,time). Names are represented by lower-case letters from a through s.

The symbol for identity: =

Socrates is wise.
Socrates is Plato's teacher.

Ws
s = p

Socrates is Plato's teacher.
You are not Socrates.
∴ You are not Plato's teacher.

Socrates is human.
You are not Socrates.
∴ You are not human.

$s = p$
$a \neq s$
$\therefore a \neq p$

Hs
$a \neq s$
$\therefore \neg Ha$

Something is coming.
$\exists x Cx$

Everything is coming.
$\forall x Cx$

Some swans are white.
$\exists x(Sx \,\&\, Wx)$

All swans are white.
$\forall x(Sx \rightarrow Wx)$

No swans are white.
$\forall x(Sx \rightarrow \neg Wx)$
$\neg\exists x(Sx \,\&\, Wx)$

Translating English into Logical Symbols

The language of predicate logic introduces two new logical symbols: the universal quantifier (∀) and the existential quantifier (∃). These can be used to translate the words something and everything:

Something is coming.
$\exists xCx$
Everything is coming.
$\forall xCx$

Every sentence in English contains a main verb. These verbs come in several kinds:

Intransitive verbs: sleeps, walks, thirsts. These take a subject only.

Transitive verbs: loves, moves,

irritates. These take a subject and a direct object.

Ditransitive verbs: gives (as in A gives B C). These take a subject, a direct object and an indirect object.

Verbs shall be translated by means of predicate letters. We shall follow these letters by one, two or three underscores, indicating whether the verb is intransitive, transitive or ditransitive.

walks ⇒ W_

likes ⇒ L_ _

gives ⇒ G_ _ _

As I described earlier, **proper names** shall be symbolized by lower-case letters from a through o (the individual constants). We will precede these constants by a question mark, to indicate that the name must be combined with a predicate in order to produce a complete sentence. An underscore shall stand for a missing name or variable, and a question mark shall stand for a missing predicate.

'John' ⇒ [? j]

Predicates consisting of 'to be' plus an adjective or common noun are also translated by means of a single predicate letter, followed by an underscore.

'is white' ⇒ W_

Consider the sentence, 'John walks'. We symbolize the name John as [? j], and the predicate '...walks' as [W?]. When these two are combined, the constant 'j' replaces the underscore in the predicate, and the predicate repaces the s question mark in the subject. The result is:

Wj

Proper names are only one kind of term. Another kind is a **noun phrase** like 'a boy' or 'all circles', in which a determiner ('a', 'all') is combined with a common noun ('boy', 'circle') or common noun phrase. A common noun phrase can be created by starting with a common noun and adding a relative clause (a clause headed by 'that', 'who', 'whom' or 'which'), and adjective or a prepositional phrase. The class of common noun phrases can be defined as follows:

1. If $\mathcal{A}$ is a common noun, then [$\mathcal{A}$] is a common noun phrase.
2. If $\mathcal{A}$ is a noun phrase, and $\mathcal{B}$ is a relative clause or a prepositional phrase, then [$\mathcal{A}$ $\mathcal{B}$] is a noun phrase.
3. If $\mathcal{A}$ is a common noun phrase, and $\mathcal{B}$ is an adjective, then [$\mathcal{B}$ $\mathcal{A}$] is a common noun phrase.

So, the following are c-noun phrases:

boy
boy in Austin
boy who lives in Austin
dark-haired boy
dark-haired boy who lives in Austin
boy who lives in a capital city

A relative clause results from replacing the subject, direct object or indirect object of a complete sentence by 'that', 'who', 'whom' or 'which' (where 'who' must always replace the subject and 'whom' the direct object). A prepositional phrase is the result of combining a preposition (like 'in', 'on', 'through', etc.) with a noun phrase. First, it is necessary to specify the translation of the **determiners** of English. The following table gives the translation of the most common determiners.

every' = any' = all' = $[\forall x(?_i\ x \rightarrow ?_j\ x)]$

a' = an' = some' = $[\exists x(?_i\ x \ \&\ ?_j\ x\ )]$

no' = none' = $[\neg\exists x(?_i\ x \ \&\ ?_j\ x)]$

only' = $[\ \forall x(\ ?_j\ x \rightarrow\ ?_j\ x)\ ]$

everybody' = anybody' =

$[\forall x(Px \rightarrow ?_i\ x)]$

somebody' = $[\exists x(Px \ \&\ ?_i\ x)]$

The subscriped letters i and j indicate that two different predicates should be used to replace the question marks. You should introduce new indices as you proceed through the English sentence. The closest predicate to the determiner should be used to replace the question mark with the lower index. Note that the crucial difference between every and only concerns the order of these indices: when translating 'every, put the first available predicate into the antecedent of the conditional, while in translating 'only' you should put the first predicate in the consequent instead. Although the variable x is used throughout this table, in fact one should introduce a new variable every time you translate a new determiner in an English sentence. So, if you've already used x, you should translate the next determiner using y instead.

Some swans are white.

[∃x($?_i$ x & $?_j$ x)] [S_] [W_]

[∃x(Sx & $?_j$ x)] [W_]

∃x(Sx & Wx)

All swans are white.

[∀x($?_i$ x → $?_j$ x)] [S_] [W_]

[∀x(Sx → $?_j$ x)] [W_]

∀x(Sx → Wx)

No swans are white.

[¬∃x($?_i$ x & $?_j$ x)] [S_] [W_]

[¬∃x(Sx → $?_j$ x)] [W_]

¬∃x(Sx & Wx)

Relative clauses consist of relative pronouns combined with predicates. The relative pronouns 'that', 'who', 'whom' and 'which' are all translated in the same way:

that' = who' = whom' = which' =

$[(?_{i}\ \_{}_{i}\ \&\ ?_{j}\ \_{}_{i}\ )]$

In this translation, the shadow question marks stand for two different predicates, but the underscores, which share the same index, stand for one individual.

Adjectives attach to common noun phrases and produce new common noun phrases. For example, the adjective 'white' can be translated as follows:

white' = $[(W\_{}_{i} \ \& \ ?_{i} \ \_{}_{i})]$

The two underscores are linked by a common index (i) and must both be replaced by the same term. The shadowed question mark should be combined with the common noun or noun phrase to which the adjective is attached.

Thus, the phrase 'a white swan' would be translated as follows:

$[\exists x(?_{i}\ x \ \& \ ?_{j}\ x)]\ [(W\_{}_{k} \ \& \ ?_{k}\ \_{}_{k})\ ]\ [S\_]$

$[\exists x(?_{i}\ x \ \& \ ?_{j}\ x)]\ [(W\_{}_{k} \ \& \ S\_{}_{k})]$

$[\exists x((Wx \ \& \ Sx) \ \& \ ?_{j}\ x)]$

At this point, we are ready to combine this noun phrase with a predicate (replacing the remaining shadowed question mark), producing a complete formula.

Prepositional phrases consist of prepositions plus noun phrases. A preposition is translated by means of a two-place predicate, a predicate letter followed by two underscores. This predicate represents the spatial or temporal relationship encoded by the preposition. For example, the translation of the predicate 'in' includes the two-place predicate I_{i}_{j}, which represents the relationship of the thing corresponding to $\_{i}$'s being inside the thing represented by $\_{j}$. The full translation of a preposition is as follows:

in' = $[(?_{i}\ \_{i}\ \&\ I\_{i}\ \_{j}\ )]$

The following translations illustrate the use of relative clauses and prepositional phrases:

All students who attend will do well.

[∀x($?_i$ x → $?_j$ x)][S?][[($?_k$ $\_{}_i$ & $?_m$ $\_{}_i$)]
[A?] [W?]
[∀x($?_i$ x → $?_j$ x)] [(S $\_\_{}_i$ & $?_m$ $\_{}_i$)]
[A?] [W?]
[∀x($?_i$ x → $?_j$ x)] [(S $\_{}_i$ & A $\_{}_i$)] [W_]
[∀x((Sx & Ax) → $?_j$ x)] [W_]
∀x((Sx & Ax) → Wx)

Some students who attend will do well.
[∃x($?_i$ x & $?_j$ x)] [S?] [[($?_k$ $\_{}_i$ & $?_m$ $\_{}_i$)]
[A_] [W_]

[∃x($?_i$ x & $?_j$ x)] [(S $\_{}_i$ & $?_m$ $\_{}_i$)]
[A_] [W_]

[∃x($?_i$ x & $?_j$ x)] [(S $\_{}_i$ & A $\_{}_i$)] [W_]
[∃x((Sx & Ax) & $?_j$ x)] [W_]
∃x((Sx & Cxc) & Wx)

A boy in Austin likes Jane.

$[\exists x(?_{i}\ x\ \&\ ?_{j}\ x\ )]\ [B{\_}][(?_{m}\ {\_}_{i}\ \&\ I{\_}_{i}\ {\_}_{j}\ )]$

$[?_{n}\ a]\ [L{\_}_{k}\ {\_}_{m}\ ]\ [?\ j]$

$[\exists x(?_{i}\ x\ \&\ ?_{j}\ x\ )]\ [B?][(?_{m}\ {\_}_{i}\ \&\ I{\_}_{i}a)]$

$[L{\_}_{k}j]$

$[\exists x(?_{i}\ x\ \&\ ?_{j}\ x\ )]\ [(B{\_}_{i}\ \&\ I{\_}_{i}\ a)][L{\_}_{k}\ j]$

$[\exists x((Bx\ \&\ Ixa)\ \&\ ?_{j}\ x\ )]\ [L{\_}_{k}\ j]$

$\forall x((Bx\ \&\ Ixa)\ \&\ Lxj)$

The following example illustrates the fact that non-restrictive relative clauses, those that are set off by commas, are to be treated quite differently.

> All Plan II students, who come to class, will do well.
> All Plan II students attend, and all Plan II students will do well.
> $(\forall x(Px \rightarrow Ax)\ \&\ \forall y(Py \rightarrow Wy))$

It is possible to have more than one determiner (and thus, more than one quantifier) in a single sentence. When translating such sentences, it can make a very material difference in what order we combine the elements. In general, when translating written text, **we should start on the right and work to the left.** Consider the following simple example:

Every boy kisses a girl.

$[\forall x(\ ?_i\ x \rightarrow\ ?_j\ x)]\ [B\_]\ [K\_{j}\ \_{k}\ ]$

$\quad [\exists y(?_k\ y \&\ ?_m\ y\ )]\ [G\_{m}]$

$[\forall x(\ ?_i\ x \rightarrow\ ?_j\ x)]\ [B?]\ [K\_{j}\ \_{k}\ ]$

$\quad [\exists y(Gy\ \&\ \ ?_m\ y\ )]$

$[\forall x(\ ?_i\ x \rightarrow\ ?_j\ x)][B\_][\exists y(Gy\ \&\ K\_{j}\ y)]$

$[\forall x(Bx \rightarrow\ ?_j\ x)]\ [\exists y(Gy\ \&\ K\_{j}\ y)]$

$\forall x(Bx \rightarrow \exists y(Gy\ \&\ Kxy))$

Pronouns (he, she, it, him, her) should be translated by means of constants or variables. In translating a pronoun in an English sentence, one should try to discern which preceding noun phrase in the sentence is the antecedent of the pronoun. Once you have identified this, you can use the corresponding constant or variable as your translation of the pronoun. For example:

Jane came, and Mike greeted her.

$(Cj \& Gmj)$

A visitor arrived, and he asked for Sally.

$\exists x((Vx \& Ax) \& A'xs)$

Every student joined a club that accepted him.

$\forall x(Sx \rightarrow \exists y(Cy \& Ayx))$

There are several special cases where the rule given above for the translation of 'a', 'an' and 'some' has to be broken. In these exceptional cases, we should translate these words as having the same meaning as 'every'.

1. **Generic "a", "an":**

A friendship is forever.
A salesman is annoying.

Although these sentence employ the indefinite article 'a', they are in fact ambiguous. They could be used to assert that there is at least one thing of the specified kind, or they could be used to assert that all or nearly all things mentioned in the subject have the characteristic expressed in the predicate. In this second case, they would be translated as:

$$\forall x(Fx \rightarrow Ex)$$
$$\forall x(Sx \rightarrow Ax)$$

2. **"A", "an", and "some" in antecedents of some conditionals, and in the subjects of some categorical statements:**

1. If you love someone, you are fortunate.
2. If you love someone, you should be good to him/her.
3. Every salesman who greets a customer will receive a bonus.
4. Every salesman who greets a customer should be polite to him/her.

In sentences 1 and 3, the words 'some' and 'a' can be translated in the standard way (employing existential quantifiers and conjunctions) without difficulty. However, if we try to do the same thing with sentences 2 and 4, we run into a serious difficulty. Simply applying our rules to these sentences would result in the following "translations":

2', $(\exists y(Py \& Lay) \rightarrow Gay)$
4'. $\forall x((Sx \& \exists y(Cy \& Gxy)) \rightarrow Pxy)$

2". $\forall y((Py \& Lay) \rightarrow Gay)$
4". $\forall x \forall y((Sx \& (Cy \& Gxy)) \rightarrow Pxy)$

"Any" tries to take a wider scope than its position in the sentence would indicate and always moves into a positive context, but it cannot leapfrog over more than one connective.

If you love everyone, you are fortunate.

$$(\forall x(Px \rightarrow Lax) \rightarrow Fa)$$

If you love anyone, you are fortunate.

$$\forall x(Px \rightarrow (Lax \rightarrow Fa))$$

I didn't catch every fish.

$$\neg \forall x(Fx \rightarrow Cbx)$$

I didn't catch any fish.

$$\forall x(Fx \rightarrow \neg Cbx)$$

I didn't catch a fish.

$\neg\exists x(Fx \,\&\, Cbx)$

If you didn't catch every fish, you should quit. $(\neg\forall x(Fx \rightarrow Cbx) \rightarrow Qb)$

If you didn't catch any fish, you should quit. $(\forall x(Fx \rightarrow \neg Cbx) \rightarrow Qb)$

Only A's are B's = All B's are A's.

Only mammals nurse their young.

$\forall x(Nx \rightarrow Mx)$

Only Republicans are happy.

$\forall x(Hx \rightarrow Rx)$

Some exercises for futher practice:

1. Nobody knows all the trouble I've seen.
2. Nobody knows any of the trouble I've seen.
3. Some of the trouble I've seen is not known by anybody.

4. Everybody knows some of the trouble I've seen.
5. Some of the trouble I've seen is known by everybody.
6. Somebody knows all the trouble I've seen.
7. Each of the troubles I've seen is known by somebody.
8. Everybody knows all of the trouble I've seen.
9. Somebody knows some of the trouble I've seen.

Pa: a is a person
Ta: a is trouble
Sab: a has seen b
Kab: a knows b
i: me

1. <u>Nobody</u> knows all the trouble I've seen.
$\neg\exists$x(Px & x knows all the trouble I've seen)

¬∃x(Px & all the trouble I've seen is known by x)

¬∃x(Px & ∀y((Ty & Siy) → y is known by x))

¬∃x(Px & ∀y((Ty & Siy) → Kxy))

2. Nobody knows any of the trouble I've seen.

∀y((Ty & Siy) → nobody knows y)

∀y((Ty & Siy) → ¬∃x x knows y)

∀y((Ty & Siy) → ¬∃x Kxy)

3. Somebody knows all the trouble I've seen.

∃x(Px & x knows all the trouble I've seen)

∃x(Px & all the trouble I've seen is known by x)

∃x(Px & ∀y((Ty & Siy) → Kxy))

4. Each of the troubles I've seen is known by somebody.

∀y((Ty & Siy) → y is known by somebody)

∀y((Ty & Siy) & <u>somebody</u> knows y)
∀y((Ty & Siy) → ∃x(Px & <u>x knows y</u>))
∀y((Ty & Siy) → ∃x(Px & Kxy))

Lab: a loves b
Pa: a is a person
a: Adam

5. <u>Everyone</u> loves someone who loves him/her.
∀x(Px → x loves someone who loves <u>him/her</u>)
∀x(Px → x loves someone who loves x)
∀x(Px → <u>someone who loves x</u> is loved by x)
∀x(Px → ∃y((Py & Lyx) & <u>y is loved by x</u>)
∀x(Px → ∃y(Py & Lyx) & Lxy))

6. <u>Everyone</u> loves someone who loves himself/herself.
∀x(Px → x loves someone who loves himself/herself)

∀x(Px → someone who loves himself/herself is loved by x)

∀x(Px → ∃y((Py & Lyy) & y is loved by x)

∀x(Px → ∃y(Py & Lyy) & Lxy))

7. Only Adam loves Adam.

Everyone who loves Adam is Adam.

∀x((Px & Lxa) → x is Adam)

∀x((Px & Lxa) → x =a)

8. Only Adam loves himself.

Everyone who loves himself/herself is Adam.

∀x((Px & Lxx) → x is Adam)

∀x((Px & Lxx) → x = a)

9. Adam loves only himself.

Everyone who is loved by Adam is himself.

Everyone who is loved by Adam is Adam.

∀x((Px & Lax) → x = a)

10. Somebody loves only Adam.
$\exists x(Px \;\&\; x \text{ loves only Adam})$
$\exists x(Px \;\&\; \text{everything loved by } x \text{ is Adam})$
$\exists x(Px \;\&\; \forall y(Lxy \rightarrow y \text{ is Adam}))$
$\exists x(Px \;\&\; \forall y(Lxy \rightarrow y = a))$

Some More Examples of Sentences with Multiple Quantifiers

Every man loves some woman.
$\forall x \exists y(Mx \rightarrow (Wy \;\&\; Lxy))$
$\forall x(Mx \rightarrow \exists y(Wy \;\&\; Lxy))$

Some woman is loved by every man.
$\exists y \forall x(Wy \;\&\; (Mx \rightarrow Lxy))$
$\exists y(Wy \;\&\; \forall x(Mx \rightarrow Lxy))$

Some woman loves every man.
$\exists y(Wy \;\&\; \forall x(Mx \rightarrow Lyx))$

Every man is loved by some woman.
$\forall x(Mx \rightarrow \exists y(Wy \;\&\; Lyx))$

Action and Event Sentences, and Adverbial Modifiers

In some cases, when translated sentences whose main verb is a verb of action or change, we must introduce an invisible term that stands for the event or action described by the verb. For example, consider the following two sentences:

John walks.
John walks slowly.

∃x(Walking(x) & By(John, x)), or ∃x(Wx & Bjx)

∃x(Walking(x) & By(John,x) & Slow(x)), or ∃x((Wx & Bjx) & Sx)

For example, we can translate 'John came late to the party' as:

∃x(Coming(x) & To(x,the party) & By(John,x) & Late(x))

We can also represent the information contained by the tenses of English statements in a similar fashion. We can introduce a special constant 'n' for the present moment, and add to any sentence in the past tense a clause of the form 'Pxn', specifying that the event x is temporally prior to the present.

An example of using function symbols in translation:

In a hierarchy, every employee tends to rise to his own level of incompetency.
Hx: x is a hierarchy.
Exy: x is an employee in y.
Rxy: x tends to rise to y
l(x): x's level of incompetency
$\forall x \forall y((Hx \,\&\, Eyx) \rightarrow Rxl(x))$

Function symbols.

In order to translate mathematical texts, we need to be able to represent functional expressions, such as addition, subtraction, and multiplication. We will do this by means of function symbols followed by a series of terms enclosed in parentheses and set off by commas. For example, to represent the complex term '2 + 3' we could use a two-place function symbol, representing addition, and two individual constants, representing 2 and 3: a(b,c), where 'b' stands for 2, 'c' for 3, and 'a' for the sum function.

Function symbols can also be used to translate certain definite descriptions in English, such as 'the capital of Texas' or 'John's mother'. These could be translated as c(e) or as m(j), where 'e' stands for Texas, 'j' for John, 'c' for the capital-of function, and 'm' for the mother-of function.

Suppose f is a 1-place, g a 2-place, h a 3-place function symbol. Then the following are terms of our formal language:

f(a) g(a,a) h(a,b,c)

f(x) g(x,a) h(x,y,z)

f(f(a))f(g(a,b)) g(f(x), h(a,x,b))

h(f(a),h(x,a,b),c)

f(g(h(a,b,c), b))

Notice that terms can be of any complexity. There are two kinds of terms, open and closed. Open terms contain variables and closed terms do not.

Formation rules

1.Any upper-case letter is a sentence.

2. Any upper-case letter followed by n names is a sentence.
3. If c and d are names, c = d is a sentence.
3. If $\mathcal{A}$ is a sentence, so is $\neg\mathcal{A}$.
4. If $\mathcal{A}$ and $\mathcal{B}$ are sentences with no variables in common, then $(\mathcal{A} \rightarrow \mathcal{B})$, $(\mathcal{A} \,\&\, \mathcal{B})$, $(\mathcal{A} \vee \mathcal{B})$, and $(\mathcal{A} \leftrightarrow \mathcal{B})$ are sentences.
5. If $\mathcal{A}$ is a sentence, c is a name, and z is a variable that does not appear in $\mathcal{A}$, then $\exists z\mathcal{A}[z/c]$ and $\forall v\mathcal{A}[z/c]$ are sentences.
6. Every sentence can be constructed by a finite number of applications of these rules.

Well—formed sentences:

$\exists xFx$ $\exists xA$ $\exists x(Fx \rightarrow A)$

$\forall x\exists yFxy$ $\forall x\forall yFy$

∀x(Fx → ∃yFxy)

(∀x∃yAxyx & ∀z∃w(Gzw → Hwwz))

¬∀x¬∃yGxy   ∃x¬(Fx → ¬∀yGxy)

Not well-formed:

∃x(Fx)   (∃xFx)   ∃x ¬ (Fx)

Fax   (Fax & ∃xGax)

∀y(Gy → Hyx)

(∃xFx & ∀xGx)   ∀x(Gx & ∃xFx)

∃xFx & ∀yGy   F∀xa

Translations of mathematical statements

Hilbert's Geometry

If two planes α, β have a point A in common, then they have at least one more point in common.

$\forall x \forall y \forall z((Ax \,\&\, Ay \,\&\, Pz \,\&\, x \neq y \,\&\, Cxz \,\&\, Cyz) \rightarrow \exists w(Pw \,\&\, Cxw \,\&\, Cyw \,\&\, w \neq z))$

There exist at least four points that do not lie in a plane.

x, y, z, w do not lie in a plane.

$\exists x \exists y \exists z \exists w((Px \,\&\, Py \,\&\, Pz \,\&\, Pw \,\&\, x \neq y \,\&\, x \neq z \,\&\, x \neq w \,\&\, y \neq z \,\&\, y \neq w \,\&\, z \neq w) \,\&\, \neg\exists u(Au \,\&\, Cux \,\&\, Cuy \,\&\, Cuz \,\&\, Cuw))$

Set Theory

If two sets have the same members, they are identical.

$\forall x \forall y((Sx \& Sy \& \forall z(z \in x \leftrightarrow z \in y)) \rightarrow x = y)$

For any set x, there is a set whose members are exactly the subsets of x.

$\forall x(Sx \rightarrow \exists y(Sy \& \forall z(z \in y \leftrightarrow z \subseteq x)))$

There is no set to which every set belongs.

There is no set x ...

...every set belongs to x

$\forall x(Sx \rightarrow \neg\forall y(Sy \rightarrow y \in x))$

$\neg\exists x(Sx \& \forall y(Sy \rightarrow y \in x))$

Every set is a subset of itself.
$\forall x(Sx \rightarrow x \subseteq x)$

No set belongs to itself.
$\forall x(Sx \rightarrow \neg x \in x)$

New Table Rules:

Instantiate Universal Quantifier

Instantiate Existential Quantifier

Alter Denied Quantification

Instantiate Universal Quantifier
From:

$\forall xFx$	$\forall x(Fx \rightarrow Gax)$	$\forall x\exists yHxyb$

We can get:

Fa	$(Fa \rightarrow Gaa)$	$\exists yHayb$
Fb	$(Fb \rightarrow Gab)$	$\exists yHbyb$

The universally quantified variable can be replaced by any constant name whatsoever.

Instantiate Existential Quantifier

Exactly the same process, with the restriction:

The name introduced by instantiating the variable on a path must be **entirely new** to the sub-tables to which the path belongs.

Consider the argument:
John is bald.
Something is thin.
∴ John is thin.

Bj
✓∃xTx
¬Tj
Tj
X

John is bald.
Everything is material.
∴ John is material.

✓1. ∀xMx
2. Bj
3. ¬Mj
4. Mj (from 1)
X

Something is thin.
Everything is material.
∴ Something is thin and material.

Alter Denied Quantifier

From $\neg\exists x \mathcal{A}$, we get $\forall x \neg \mathcal{A}$.

From $\neg\forall x \mathcal{A}$, we get $\exists x \neg \mathcal{A}$.

If it's false that something is red, then everything is not red.
If it's false that everything is red, then something is not red.

Numerical statements:
There is at least one F. $\exists xFx$

There are at least two F's.
$\exists x(Fx\&Fy\&\neg x=y)$

There are at least three Fs.
$\exists x\exists y\exists z(Fx\&Fy\&Fz\&\neg x=y \&\neg y=z \& \neg x=z)$

There is at most one F.
$\forall x\forall y((Fx\&Fy) \rightarrow x=y)$

There are at most two Fs.
$\forall x\forall y\forall z((Fx\&Fy\&Fz)\rightarrow(x=y\vee y=z\vee x=z)$

There is exactly one F.
$\exists x(Fx \& \forall y(Fy \rightarrow x=y))$

Mab: a is more … than b

To say: a is the most…,
$\forall x(\neg x=a \rightarrow Max)$
To say: a is the least…,
$\forall x(\neg x=a \rightarrow Mxa)$

Enthymemes in English

An enthymeme is an argument in which one or more premises -- consisting of universal generalizations -- have been left unstated.

Persons of genius are, by definition, more individual than any other people -- less capable, consequently, of fitting themselves, without hurtful compression, into any of the small number of molds which society provides.

J. S. Mill

$G\alpha$: α is a genius
$I\alpha\beta$: α is more individual than β
$F\alpha\beta\gamma$: α is more capable of fitting himself without hurtful compression into any of the small number of molds β provides than is γ.
a: society
$\forall x \forall y((Gx \& \neg Gy) \to Ixy)$
$\therefore \forall x \forall y((Gx \& \neg Gy) \to Fyax)$

Missing premise:

Sir, when a man is tired of London, he is tired of life, for in London is to be found all that life affords. [Samuel Johnson]

l: London

b: life

$T\alpha\beta$: α is tired of β

$A\alpha\beta$: α affords β (or β is to be found in α)

$\forall x(Abx \rightarrow Alx)$

$\therefore \forall x(Txl \rightarrow Txb)$

Missing premises:

$\forall x \forall y(\forall z(Ayz \rightarrow Txz) \rightarrow Txy))$

For all x and y, x is tired of y only if x is tired of everything y affords.

$\forall x \forall y(Px \rightarrow (Txy \rightarrow \forall z(Ayz \rightarrow Txz)))$

For all x and y, x is tired of y if x is tired of everything y affords.

If individuals live only 70 years, then a state, or a nation, or a civilization, which may last for a thousand years, is more important than an individual. But if Christianity is true, then the individual is not only more important, but incomparably more important, for he is everlasting, and the life of a state or a civilization, compared with his, is only a moment.

C. S. Lewis

Iα: α is an individual human
Sα: α is a state
Cα: α is a civilization
Mα: α lives only 70 years (or so)
Lα: α lives for centuries or millenia
Eα: α lives forever
D$\alpha\beta$: α lives longer than β
V$\alpha\beta$: α is more important than β
A: Christianity is true

Premises:
$(A \rightarrow \forall x(Hx \rightarrow Ex))$
$(\forall x(Hx \rightarrow Ex) \rightarrow$
$\forall y \forall z((Hy \& (Sz \vee Cz)) \rightarrow Dyz))$

Conclusion:
$(A \rightarrow \forall x \forall y((Hx \& (Sy \vee Cy)) \rightarrow Vxy))$

Unstated premise:
$\forall x \forall y(Dxy \rightarrow Vxy)$

Paralogism from Plato's *Euthydemos*

Learners do not know what they are learning.
Those who do not know something are ignorant.
∴ Learners are ignorant.

$\forall x \forall y(Lxy \rightarrow \neg Kxy)$
$\forall x \forall y(\neg Kxy \rightarrow Ixy)$
$\therefore \ \forall x \forall y(Lxy \rightarrow Ixy)$

Learners come to know what they are learning.
Those who come to know something are not ignorant.
∴ Learners are not ignorant.

$\forall x x \forall y(Lxy \rightarrow K'xy)$
$\forall x \forall y(K'xy \rightarrow \neg Ixy)$
∴ $\forall x \forall y(Lxy \rightarrow \neg Ixy)$

Inconsistency results if we add $\exists x \exists y$ Lxy

A poem is nothing but letters.
Those who learn a poem already know all the letters.
∴ Those who learn a poem learn what they already know.

$\forall x(Px \rightarrow Ay)$
$\forall x(\exists y(Lxy \,\&\, Py) \rightarrow \forall z(Az \rightarrow Kxz))$
$\therefore \forall x(\exists y(Lxy \,\&\, Py) \rightarrow \exists z(Lxz \,\&\, Kxz))$
or $\forall x \forall y((Lxy \,\&\, Py) \rightarrow Kxy)$

Distinguish between: 'is a letter'.
and 'is (a collection of) letters'.
Ay: y is a letters (like alpha, beta, gamma)
A'y: y is composed by letters (like poems, histories, etc.)

You want Cleanthes to be a wise person.
The only wise person is Socrates.
∴ You want Cleanthes to be Socrates.
∴ You want Cleanthes not to be Cleanthes any more.
∴ You want Cleanthes to die.

W: ______wants _______ to be ______

$\exists x(Sx \ \& \ Wacx)$
$\forall x(Sx \rightarrow x = s)$
$\therefore$ Wacs
$c \neq s$
$\therefore \neg$Wacc
$\therefore$ Wacd

Verbs of attitude takes states of affairs as objects.
$H\alpha\beta\gamma$: in state α, it holds that β has property γ
s': property of wisdom
s": the property of being identical to Socrates
e: the actual world
$\exists x(Wax \ \& \ Hxcs')$
$\forall x(Hexs' \rightarrow s = s')$, or $\forall x(Hexs' \rightarrow Hexs'')$
$\therefore \exists x(Wax \ \& \ Hxcs'')$

Your dog is a father.
Your dog is yours.
$\therefore$ Your dog is your father.

When someone tells a lie, he speaks of something.
Whatever someone speaks of is real.
When someone speaks of something real, he speaks truly.
∴ When someone tells a lie, he speaks truly.

Rights of passage

The following are equivalent:
$\forall x \forall y(Fx \rightarrow Gxy)$ $\forall x(Fx \rightarrow \forall y Gxy)$

So are these:
$\forall x \exists y(Fx \rightarrow Gxy)$ $\forall x(Fx \rightarrow \exists y Gxy)$

And these:
$\forall x \forall y(Gxy \rightarrow Fx)$ $\forall x(\exists y Gxy \rightarrow Fx)$
These too:
$\forall x \exists y(Gxy \rightarrow Fx)$ $\forall x(\forall y Gxy \rightarrow Fx)$

The following pairs are **not** equivalent:
$\forall x \forall y(Gxy \rightarrow Fx)$ $\forall x(\forall y Gxy \rightarrow Fx)$

$\forall x \exists y(Gxy \rightarrow Fx)$ $\forall x(\exists y Gxy \rightarrow Fx)$

Moving a quantifer through a negative context changes its polarity ($\forall$ to $\exists$ and vice versa).

$\forall x(\exists y Lxy \rightarrow \neg Ix)$ equiv to:

$\forall x \forall y(Lxy \rightarrow \neg Ix)$

$\forall x(\forall z Lxz \rightarrow Ix)$ equiv. to

$\forall x \exists z\, (Lxz \rightarrow Ix)$

Moving through a positive context, no change:

All students and all teachers have a conference.
$\forall x(Sx \rightarrow \forall y(Ty \rightarrow \exists z Czxy))$

$\forall x \forall y \exists z(Sx \rightarrow (Ty \rightarrow Czxy))$

$\forall x \forall y \exists z((Sx \& Ty) \rightarrow Czxy)$