CURVE SKETCHING

This is a handout that will help you systematically sketch functions on a coordinate plane. This handout also contains definitions of relevant terms needed for curve sketching. Another handout available on the handouts wall has 3 sample problems worked out completely.

ASYMPTOTES:

This handout will discuss three kinds of asymptotes: vertical, horizontal, and slant.

VERTICAL ASYMPTOTES

We define the line $x = c$ as a **vertical asymptote** of the graph of $f(x)$ iff (if and only if) $f(x)$ approaches infinity as x approaches c from the right or left.

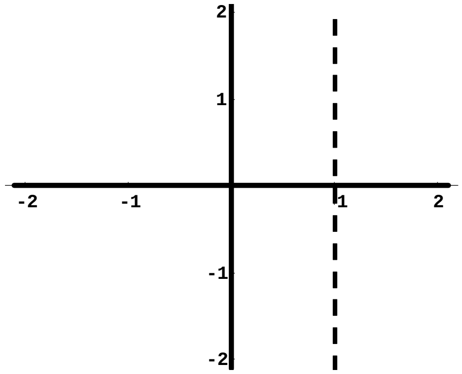
The concept of an asymptote is best illustrated in the following example:

Take the function $f(x) = \frac{2x}{1-x}$

Here, we can see that x cannot take the value of 1, otherwise, $f(x)$ would be undefined. Also:

$$\lim_{x\to 1+} f(x) = -\infty \quad \text{and} \quad \lim_{x\to 1-} f(x) = +\infty$$

In this case, we call the line $x = 1$ a **vertical asymptote** of $f(x)$.



Vertical Asymptote: $x = 1$

The fact that $f(x)$ is undefined at $x = 1$ is not enough to conclude that we have a vertical asymptote. The function must also approach infinity or negative infinity as x approaches the value at which $f(x)$ is undefined.

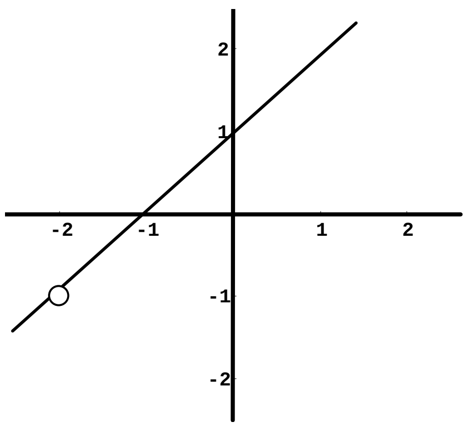
Consider the following problem:

$$f(x) = \frac{x^2 + 3x + 2}{x + 2}$$

The function $f(x)$ is **undefined at** $x = -2$ but we do not have an asymptote. Notice the following:

$$\lim_{x \to -2} \frac{x^2 + 3x + 2}{x + 2} = \lim_{x \to -2} \frac{(x+2)(x+1)}{x+2} = \lim_{x \to -2} x + 1 = -2 + 1 = -1$$

We conclude that $f(x)$ approaches -1 as x approaches -2. This function has a "hole" (removable discontinuity), not an asymptote, at the value for which $f(x)$ is undefined.



$$f(x) = \frac{x^2 + 3x + 2}{x + 2}$$

In general, you may note that a rational function (a function that is the quotient of two polynomial functions), that is fully reduced (the numerator and denominator cannot be factored such that they share a common factor which can be "cancelled") will have a vertical asymptote at each x-value that causes the denominator to evaluate to zero.

Once again, in order to have an asymptote at $x = c$, $f(x)$ must have a discontinuity at c **and** $f(x)$ must approach positive or negative infinity, as x approaches c from the left or the right.

HORIZONTAL ASYMPTOTES

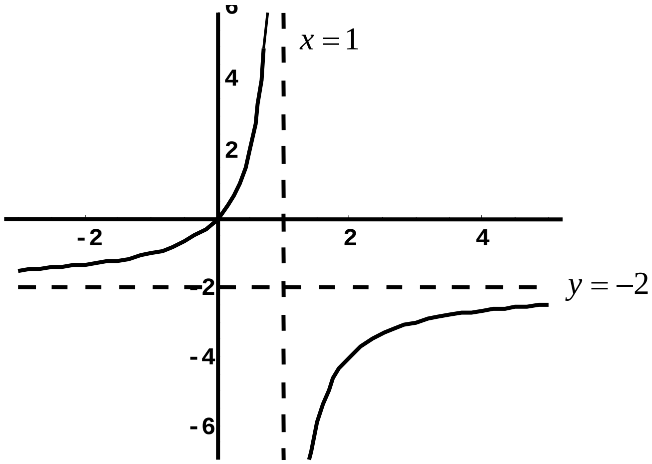
We define the line $y = L$ as a **horizontal asymptote** of the graph of $f(x)$ iff $f(x)$ approaches L as x approaches infinity.

For the function $f(x) = \frac{2x}{1 - x}$ the line $y = -2$ is the horizontal asymptote of the graph of $f(x)$.

The following limit shows why this is true:

$$\lim_{x\to\infty}\frac{2x}{1-x}=\lim_{x\to\infty}\frac{2}{\frac{1}{x}-1}=\frac{2}{-1}=-2 \quad \textbf{and} \quad \lim_{x\to-\infty}\frac{2x}{1-x}=\lim_{x\to-\infty}\frac{2}{\frac{1}{x}-1}=\frac{2}{-1}=-2$$

When x approaches infinity, $f(x)$ approaches the line $y=-2$, and when x approaches negative infinity, $f(x)$ also approaches the line $y=-2$.



A quick way to determine the position of the horizontal asymptote of a rational function (having no common factors) is with the following method. Look at the highest degree in the numerator and the highest degree in the denominator.

- If the highest degree is in the denominator, then the horizontal asymptote is $y=0$.

- If the highest degree in the numerator and the highest degree in the denominator are equal, then the horizontal asymptote is the ratio of the coefficient of the highest degree term in the numerator to the coefficient of the highest degree term in the denominator. (In our previous example, $y=\frac{2x}{1-x}$, the highest degree in the numerator is 1 and the highest degree in the denominator is 1. The ratio of highest degree term coefficients is $\frac{2}{-1}$. So the horizontal asymptote is $y=-2$.)

- If the highest degree in the numerator is one degree larger than the highest degree in the denominator, then the function has a slant asymptote.

- If the highest degree in the numerator is more than one degree larger than the highest degree in the denominator, then the function has no horizontal or slant asymptote.

SLANT ASYMPTOTES

If the highest degree in the numerator of a rational function (having no common factors) is one degree larger than the highest degree in the denominator, we say that the function has a **slant asymptote**.

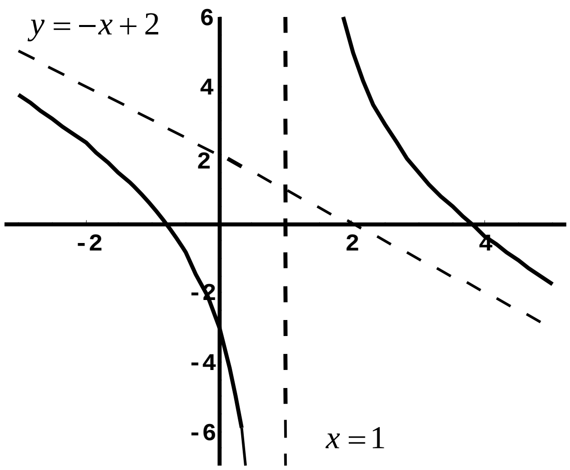
To determine the asymptote, rewrite the function in terms of a polynomial + another rational function. For example, let

$$f(x)=\frac{-x^2+3x+3}{x-1}.$$

Dividing the numerator by $x-1$ (see "Synthetic Division" handout), we get

$$f(x)=-x+2+\frac{5}{x-1}.$$

Since the fraction $\frac{5}{x-1}$ approaches 0 as x approaches infinity and negative infinity, the function $f(x)$ approaches the line $y=-x+2$ as x approaches infinity and negative infinity. $y=-x+2$ is called a slant asymptote.



$$f(x)=\frac{-x^2+3x+3}{x-1}$$

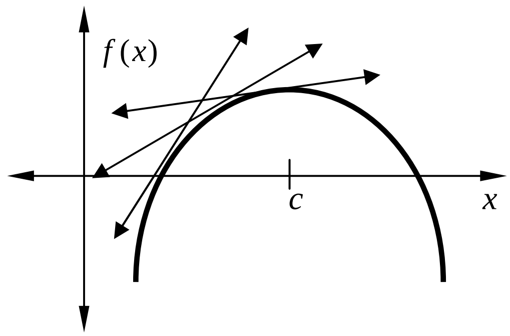
Some Remarks about Functions with Asymptotes:

- Vertical asymptotes are NEVER crossed by $f(x)$. However, the graph of $f(x)$ may sometimes cross a horizontal or slant asymptote.
- Asymptotes help determine the shape of the graph.
- Polynomial functions never have asymptotes.

INCREASING AND DECREASING:

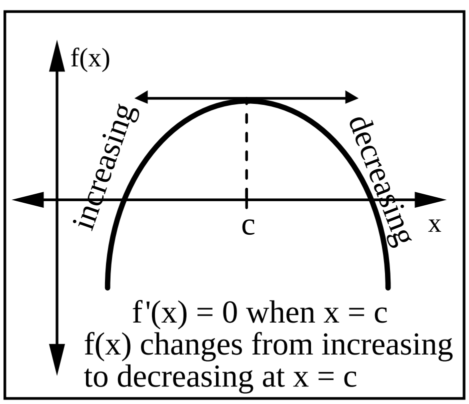
A function is increasing over an interval if, when tracing the graph from left to right, the graph is going up. Likewise, a function is decreasing over an interval when the graph is going down (when tracing the graph from left to right).

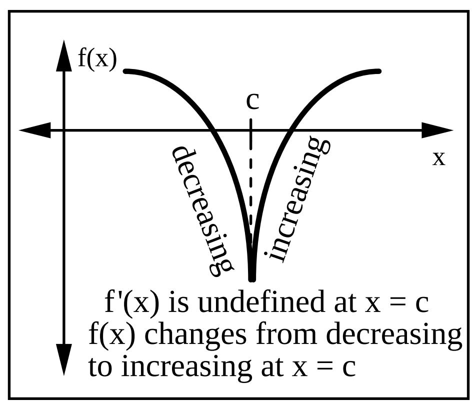
Mathematically, the function $f(x)$ is **increasing** when $f'(x)$ (the slope of the tangent line to the curve) **is positive** and $f(x)$ is **decreasing** when $f'(x)$ **is negative**.



Slopes of the tangent lines are positive when $x < c$, so $f(x)$ is increasing when $x < c$.

A function $f(x)$ can change from increasing to decreasing (or vice versa) at values where $f'(x) = 0$ or $f'(x)$ is undefined.





To find where a function is increasing and where it is decreasing:

1. Compute $f'(x)$.
2. Determine the value(s) of x where $f'(x) = 0$ or where $f'(x)$ is undefined.
3. Order the values found in (2) in increasing order and plot them on a number line.
4. For every interval between two consecutive values in (3), choose a test value in that interval.
5. Determine the value of $f'(x)$ at the test value.
6. If $f'(x) > 0$ at the test value, then $f(x)$ is increasing on that interval. If $f'(x) < 0$ at the test value, then $f(x)$ is decreasing on that interval.

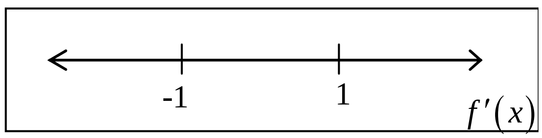
<u>For example</u>: Let $f(x) = 3x^3 - 9x$. Determine where $f(x)$ is increasing and where it is decreasing.

Taking the first derivative and setting it equal to zero, we obtain:

$$f'(x) = 9x^2 - 9 = 0$$
$$9(x^2 - 1) = 0$$
$$9(x+1)(x-1) = 0 \text{ ---->} \quad x = 1 \text{ and } x = -1$$

$f'(x)$ is defined everywhere, so we have only 2 values where $f(x)$ can change from increasing to decreasing: $x = 1$ and $x = -1$.

Order the values found above on a number line as follows:



The intervals we need to test are (-∞,-1), (-1,1) and (1,∞):

A) For the interval **(-∞,-1)**, we will choose $x = -2$ as our test value.

$$f'(-2) = 9(-2)^2 - 9 = 36 - 9 = 27$$

Since $f'(-2) > 0$, we know $f(x)$ is **increasing** on the interval (-∞,-1).

B) For the interval **(-1,1)**, we choose $x = 0$ as our test value.

$$f'(0) = 9(0)^2 - 9 = -9$$

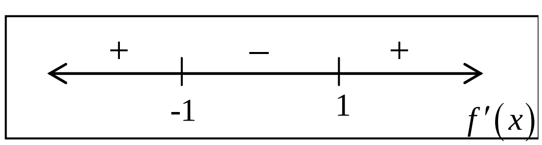
Therefore, since $f'(0) < 0$, $f(x)$ is **decreasing** on the interval (-1,1).

C) For the interval **(1,∞)**, we choose $x = 2$ as our test value.

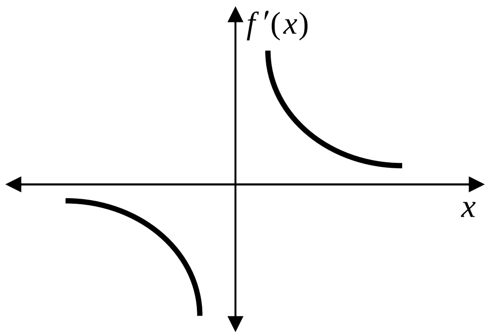
$$f'(2) = 9(2)^2 - 9 = 36 - 9 = 27$$

Again, $f(x)$ is **increasing** on the interval (1,∞), because $f'(2) > 0$.

Relabeling our number line we have the following:



Remember that values where $f'(x) = 0$ or $f'(x)$ is undefined are only <u>potential</u> places where the graph can change from increasing to decreasing (or vice versa). It is possible, however, that the function may not change at those values.



$f(x)$ is undefined at $x = 0$
$f(x)$ is decreasing when $x < 0$
$f(x)$ is decreasing when $x > 0$

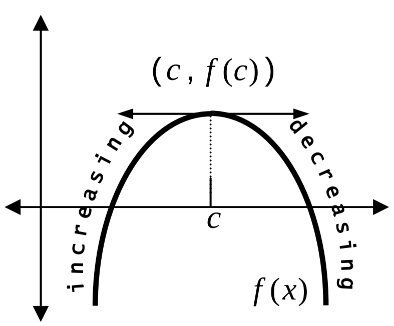
CRITICAL/EXTREME POINT:

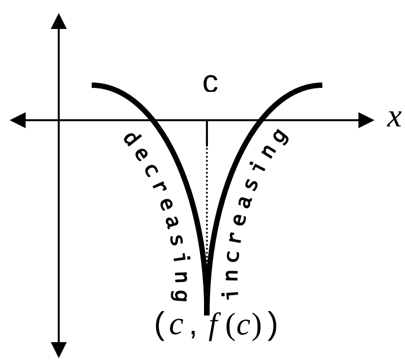
Another important concept needed in curve sketching is that of a **critical point**. If $x = c$ is in the domain of $f(x)$ and either $f'(c) = 0$ or $f'(c)$ is not defined, then $x = c$ is called a **critical value** of the function $f(x)$, and $(c, f(c))$ is called a critical point. A critical point may be a maximum point, minimum point, or neither.

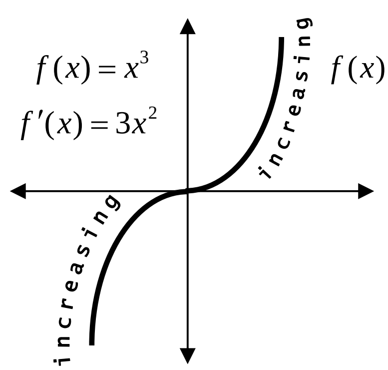
A relative (or local) **maximum point** is a critical point where the function changes from increasing to decreasing.

A relative (or local) **minimum point** is a critical point where the function changes from decreasing to increasing.

The critical point is **neither** a maximum nor a minimum if the function does not change from increasing to decreasing (or vice versa) at the critical point.







- $f(c)$ is defined and $f'(c) = 0$ so $(c, f(c))$ is a critical point.
- $f(x)$ is increasing before $x = c$ and decreasing after $x = c$ so $(c, f(c))$ is a

- $f(c)$ is defined and $f'(c)$ is undefined so (c,f(c)) is a critical point.
- $f(x)$ is decreasing before $x = c$ and increasing after $x = c$ so $(c, f(c))$ is a relative minimum.

- $f(0)$ is defined and $f(0) = 0$ so (0,0) is a critical point.
- $f(x)$ is increasing before $x = 0$ and increasing after $x = 0$ so $(c, f(c))$ is neither a max nor a min.

To locate the critical points on the graph:

1. Take the first derivative of the function and determine the values $x = c$ where $f'(c) = 0$ or $f'(c)$ is undefined.
2. If c is in the domain of $f(x)$, then $(c, f(c))$ is a critical point.

<u>For example</u>:

$$f(x) = 3x^3 - 9x$$

Earlier we found that $f'(x) = 9x^2 - 9 = 0$, when $x = -1$ and $x = 1$, and $f'(x)$ is defined everywhere. Since $f(x)$ is defined for both $x = -1$ and $x = 1$, we have found two critical values.

Substituting these values into our original function, we find that

$$f(1) = 3(1)^3 - 9(1) \qquad \text{AND} \qquad f(-1) = 3(-1)^3 - 9(-1)$$
$$= 3 - 9 \qquad\qquad\qquad = -3 + 9$$
$$y = -6 \qquad\qquad\qquad y = 6$$

Thus, we have found **critical points** at **(1,-6)** and **(-1,6)**.

You can determine whether these points are local maximum points, local minimum points, or neither, using either the first derivative test or the second derivative test (the second derivative test will be explained in the next section).

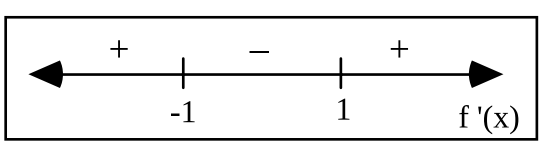
<u>FIRST DERIVATIVE TEST</u>

1. Determine where the function is increasing or decreasing.
2. If the function is increasing before the critical value and decreasing after the critical value, then the critical point is a local maximum. If the function is decreasing before the critical value and increasing after, the point is a local minimum. Otherwise, the critical point is neither a maximum nor a minimum.

<u>For example</u>: Earlier we found critical points for $f(x) = 3x^3 - 9x$ at (1,-6) and (-1,6). To determine whether these points are local maximums or minimums, use the first derivative test.

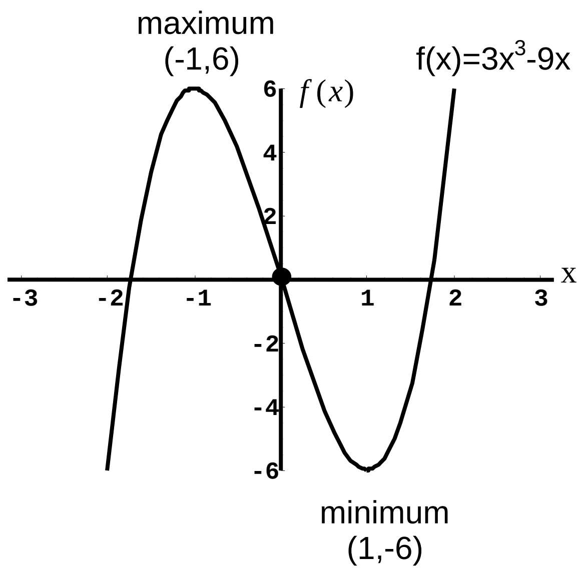
First, determine where $f(x) = 3x^3 - 9x$ is increasing and decreasing.
The critical values found above were $x = 1$ and $x = -1$.

From our previous example we found that $f(x)$ is increasing on $(-\infty,1)$, decreasing on (-1,1), and increasing on $(1,\infty)$. Thus, we had the following number line:



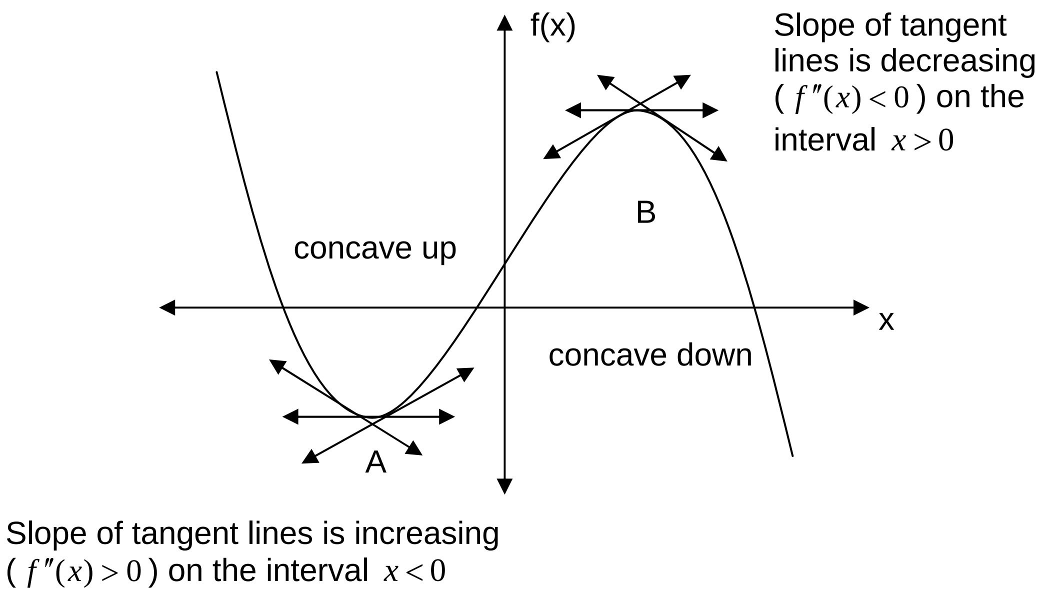
At the critical value $x = -1$, the function changes from increasing to decreasing. Therefore, **(-1,6)** is a local **maximum**.

At the critical value $x = 1$, the function changes from decreasing to increasing. Therefore, **(1,-6)** is a local **minimum**.



CONCAVITY:

Concavity describes the general "cupping" of a function at a particular point or interval. When the slope of the tangent to the curve is increasing over an interval ($f'(x)$ is increasing - i.e. $f''(x) > 0$), the function is concave up. When the slope of the tangent to the curve is decreasing over an interval ($f'(x)$ is decreasing, i.e. $f''(x) < 0$), the function is concave down.



Notice that the point A is a critical point (since the slope of the tangent line is 0 at A) and $f(x)$ is concave up at A. We can see that A is a local minimum. Also, the point B is a critical point (since the slope of the tangent line is 0 at B) and $f(x)$ is concave down at B. We can see that B is a local maximum.

Concavity can help us determine if a critical point is a local maximum or a minimum. The following is the second method for determining whether a critical point is a local maximum or a minimum.

SECOND DERIVATIVE TEST

If $(c, f(c))$ is a critical point, then:

1. If $f''(c) < 0$, the function is concave down at that point and thus $(c, f(c))$ is a local **maximum point**.
2. If $f''(c) > 0$, the function is concave up at that point and thus $(c, f(c))$ is a local **minimum point**.
3 If $f''(c) = 0$, then the second derivative test **fails** to determine if the point is a local maximum or a minimum. In this case, the first derivative test mentioned earlier should be used.

Example: Let's go back to the function $f(x) = 3x^3 - 9x$

$$f'(x) = 9x^2 - 9$$

$$f''(x) = 18x$$

Our critical points were **(-1,6)** and **(1,-6)**.
Using the second derivative test we obtain the following:

$f''(-1) = -18$ Since $f''(-1) < 0$, the function is concave down at **(-1,6)**, thus **(-1,6)** is a local **maximum**.

$f''(1) = 18$ Since $f''(1) > 0$, the function is concave up at **(1,-6)**, thus **(1,-6)** is a local **minimum**.

We see that we get the same results using the Second Derivative Test as we do using the First Derivative Test.

INFLECTION POINTS:

An inflection point is a point on a graph (in the domain of $f(x)$) where concavity changes from concave up to concave down, or vice-versa. Concavity can change at values where $f''(x) = 0$ or $f''(x)$ is undefined.

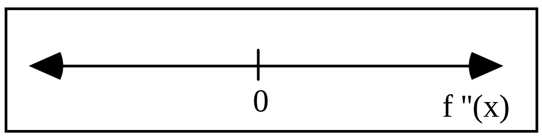
To find inflection points:

1. Determine where the function is concave up and where it is concave down:
   a) Determine the value(s) of x where $f''(x)=0$ or $f''(x)$ is undefined.
   b) Order the values found above in increasing order and plot them on a number line.
   c) For every interval between two consecutive values, choose a test value in that interval.
   d) Determine the value of $f''(x)$ at the test value.
   e) If $f''(x)>0$ at the test value, then $f(x)$ is concave up on that interval. If $f''(x)<0$ at the test point, then $f(x)$ is concave down on that interval.

2. If the function changes from concave up to concave down (or vice-versa) at $x=c$ and $f(c)$ is defined, then (c , $f(c)$) is an inflection point.

<u>Example</u>: Earlier we found that the second derivative of $f(x)=3x^3-9x$ was $f''(x)=18x$. Thus $f''(x)=0$ when $x=0$, and $f''(x)$ is defined everywhere.

Plotting this on a number line we get:



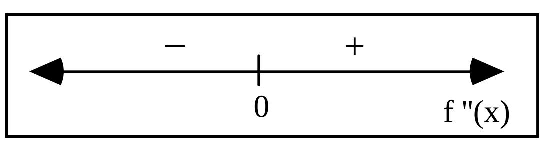
For the interval (-∞,0), choose $x=-1$ to be our test point: $f''(-1)=-18$

Since $f''(-1)<0$, $f(x)$ is **concave down** on the interval (-∞,0).

For the interval (0, ∞), choose $x=1$ to be our test point: $f''(1)=18$

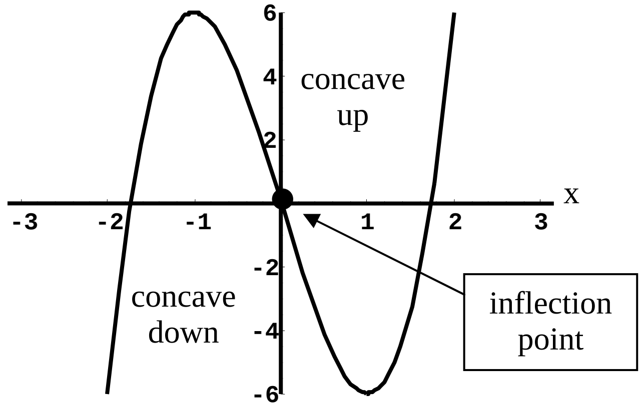
Since $f''(1)>0$, $f(x)$ is **concave up** on the interval (0, ∞).

Labeling our number line we get:



Since $f(x)$ changes from concave down to concave up at $x=0$ and $f(x)$ is defined at $x=0$, the point $(0, f(0))=(0,0)$ is an inflection point.

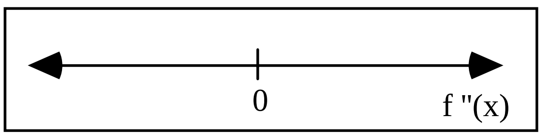
$f(x)$ f(x)=3x^3-9x



Remember that values where $f''(x)=0$ or $f''(x)$ is undefined are only potential places where the graph can change concavity. It is possible, however, that the function may not change concavity at those values.

For example, consider $f(x)=x^4$. Then $f'(x)=4x^3$ and $f''(x)=12x^2$, hence $f''(x)=0$ when $x=0$.

Plotting this on a number line we get:



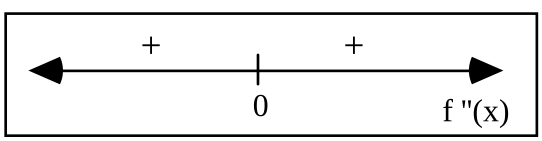
For the interval (-∞,0), choose $x=-1$ to be our test point: $f''(-1)=12$

Since $f''(-1)>0$, $f(x)$ is **concave up** on the interval (-∞,0).

For the interval (0,∞), choose $x=1$ to be our test point: $f''(1)=12$

Since $f''(1)>0$, $f(x)$ is **concave up** on the interval (0, ∞).

Labeling our number line we get:



In this example $f(x)=x^4$ is concave up when $x<0$ and concave up when $x>0$. Concavity did not change at $x=0$, so the point (0,0) is not an inflection point.