CURVE SKETCHING EXAMPLES

This handout contains three curve sketching problems worked out completely.

<u>Sample Problem #1</u>:

$$f(x) = x^3 - 6x^2 + 9x + 1$$

1. Look for any **asymptotes:** Polynomial functions do not have asymptotes.

a) Vertical: No vertical asymptotes because $f(x)$ is continuous for all x.
b) Horizontal: No horizontal asymptotes because $f(x)$ is unbounded as $x \to \pm\infty$.

2. **Intercepts:**

a) y-intercepts: $f(0) = 1$ **y-intercept:** $(0,1)$

b) x-intercepts: $0 = x^3 - 6x^2 + 9x + 1$ (difficult to find – skip)

3. **Increasing/Decreasing:**

a) Take the first derivative: $f'(x) = 3x^2 - 12x + 9$

b) Set it equal to zero: $3x^2 - 12x + 9 = 0$

c) Solve for x: $3(x^2 - 4x + 3) = 0$
$3(x-1)(x-3) = 0$
$x = 1$ and $x = 3$

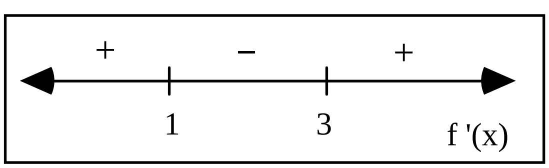
d) Where is $f'(x)$ undefined? Nowhere

e) Sign analysis: Plot the numbers found above on a number line. Choose test values for each interval created and evaluate the first derivative:

$f'(0) = 3(0)^2 - 12(0) + 9 = 9$ Positive $\Rightarrow$ $f(x)$ **is increasing on** $(-\infty,1)$.

$f'(2) = 3(2)^2 - 12(2) + 9 = -3$ Negative $\Rightarrow$ $f(x)$ **is decreasing on** $(1,3)$.

$f'(4) = 3(4)^2 - 12(4) + 9 = 9$ Positive $\Rightarrow$ $f(x)$ **is increasing on** $(3,\infty)$.



4. **Extrema:**

a) For which values of x (found above in part 3) is $f(x)$ defined? $x=1$ and $x=3$

b) Find corresponding values of y: $f(1)=(1)^3-6(1)^2+9(1)+1=5$
$f(3)=(3)^3-6(3)^2+9(3)+1=1$

c) Critical points: $(1,5)$ and $(3,1)$

d) Extrema: $f'(x)$ changes from positive to negative at $x=1$ and from negative to positive at $x=3$ so $(1,5)$ and $(3,1)$ are local extrema of f .

Note: Values of x corresponding to local extrema of f must:

- Be critical values of the first derivative – values at which $f'(x)$ equals zero or is undefined,
- Lie in the domain of f , and
- Be values at which the sign of the first derivative changes.

5. Test extrema for local **max/mins**:

SECOND DERIVATIVE TEST

a) Take the second derivative: $f''(x)=6x-12$

b) Substitute x-coordinates of extrema: $f''(1)=6(1)-12=-6$ (negative $\Rightarrow$ local max)
$f''(3)=6(3)-12=6$ (positive $\Rightarrow$ local min)

c) Label your point(s): **local max: (1,5)** **local min: (3,1)**

or :

FIRST DERIVATIVE TEST

a) $f(x)$ is increasing before $x=1$ and decreasing after $x=1$: **(1,5)** is a **maximum**

b) $f(x)$ is decreasing before $x=3$ and increasing after $x=3$: **(3,1)** is a **minimum**

6. **Concave up/Concave down:**

a) Set $f''(x)$ equal to zero: $6x-12=0$

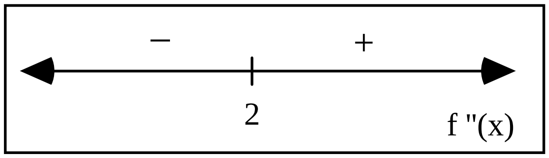
b) Solve for x: $x=2$

c) Where is $f''(x)$ undefined? Nowhere

d) Sign analysis: Plot the numbers found above on a number line. Choose test values for each interval created and evaluate the second derivative.

$f''(1) = 6(1) - 12 = -6 < 0 \Rightarrow f(x)$ **is concave down on** $(-\infty, 2)$.

$f''(3) = 6(3) - 12 = 6 > 0 \Rightarrow f(x)$ **is concave up on** $(2, \infty)$.



7. Find any **inflection points:**

a) For which values of x (found in 6) is $f(x)$ defined? $x = 2$

b) Find corresponding value of y: $f(2) = (2)^3 - 6(2)^2 + 9(2) + 1 = 3$

c) $f(x)$ changes from concave up to concave down at $x = 2$, so $(2,3)$ is an inflection point. **Inflection point: (2,3)**

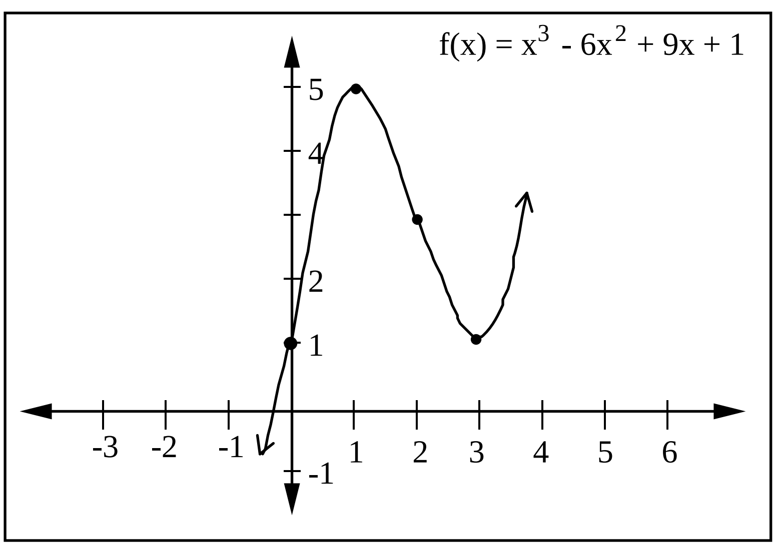
Note: Values of x corresponding to inflection points of f must:

- be critical values of the second derivative – values at which $f''(x)$ equals zero or is undefined,
- lie in the domain of f, and
- be values at which the sign of the second derivative changes.

8. Note in a chart your points obtained:

x	y	
1	5	(maximum point)
3	1	(minimum point)
2	3	(inflection point)
0	1	(y - intercept)

9. Plot all points on the coordinate plane and sketch in the rest of the graph. Be sure to include all maximum points, minimum points, and inflection points:



Sample Problem #2:

$$f(x) = 3x^5 - 5x^3$$

1. Look for any **asymptotes:**

a) Vertical: No vertical asymptotes as $f(x)$ is continuous for all x.

b) Horizontal: No horizontal asymptotes as $f(x)$ is unbounded as $x \to \pm\infty$.

2. **Intercepts:**

a) y-intercepts: $f(0) = 0$ **y-intercept:** $(0,0)$

b) x-intercepts: $3x^5 - 5x^3 = 0$

$x^3(3x^2 - 5) = 0$

$x = 0$ and $x = \pm\sqrt{\frac{5}{3}}$ **x-intercepts:** $(0,0)$, $\left(\sqrt{\frac{5}{3}},0\right)$, $\left(-\sqrt{\frac{5}{3}},0\right)$

3. **Increasing/Decreasing:**

a) Take the first derivative: $f'(x) = 15x^4 - 15x^2$

b) Set it equal to zero: $15x^4 - 15x^2 = 0$

c) Solve for x: $15x^2(x^2 - 1) = 0$

$x = 0$, $x = 1$, and $x = -1$

d) Where is $f'(x)$ undefined? Nowhere

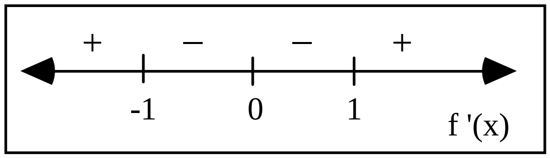
e) Sign analysis: Plot the numbers found above on a number line. Choose test values for each interval created and evaluate the first derivative.

$f'(-2) = 15(-2)^4 - 15(-2)^2 = 180$ Positive $\Rightarrow$ $f(x)$ **is increasing on** $(-\infty,-1)$.

$f'(-\frac{1}{2}) = 15(-\frac{1}{2})^4 - 15(-\frac{1}{2})^2 = -\frac{45}{16}$ Negative $\Rightarrow$ $f(x)$ **is decreasing on** $(-1,0)$.

$f'(\frac{1}{2}) = 15(\frac{1}{2})^4 - 15(\frac{1}{2})^2 = -\frac{45}{16}$ Negative $\Rightarrow$ $f(x)$ **is decreasing on** $(0,1)$.

$f'(2) = 15(2)^4 - 15(2)^2 = 180$ Positive $\Rightarrow$ $f(x)$ **is increasing on** $(1,-\infty)$.



4. **Extrema:**

a) For which values of x (found above in 3) is $f(x)$ defined? $x=-1$**,** $x=0$**,** and $x=1$

b) Find corresponding values of y:

$$f(0)=3(0)^5-5(0)^3=0$$
$$f(1)=3(1)^5-5(1)^3=-2$$

c) Critical points: **(-1,2), (0,0),** and **(1,-2)**

d) Extrema: $f'(x)$ changes from positive to negative at $x=-1$ and from negative to positive at $x=1$ so **(-1,2)** and **(1,-2)** are local extrema of f. However, $f'(x)$ does not change sign at $x=0$, so **(0,0)** is not an extremum of f.

5. Test extrema for local **max/mins:**

<u>SECOND DERIVATIVE TEST</u>

a) Take the second derivative: $f''(x)=60x^3-30x$

b) Evaluate at x-coordinates of extrema:

$f''(-1)=60(-1)^3-30(-1)=-30$ (negative $\Rightarrow$ local max)
$f''(0)=60(0)^3-30(0)=0$ (zero $\Rightarrow$ test fails, must use the first derivative test)
$f''(1)=60(1)^3-30(1)=30$ (positive $\Rightarrow$ local min)

c) Label your points: **(-1,2) : local max**
(1,-2) : local min
(0,0) : unknown at this time

or:

<u>FIRST DERIVATIVE TEST</u>

a) $f(x)$ is increasing before $x=-1$ and decreasing after $x=-1$.
(-1,2) is a local maximum.

b) $f(x)$ is decreasing before $x=0$ and decreasing after $x=0$.
(0,0) is neither a max nor a min.

c) $f(x)$ is decreasing before $x=1$ and increasing after $x=1$.
(1,-2) is a local minimum.

6. **Concave up/Concave down:**

a) Set $f''(x)$ equal to zero: $60x^3 - 30x = 0$

b) Solve for x: $30x(2x^2 - 1) = 0$ $x = 0$, $x = \pm\frac{\sqrt{2}}{2}$

c) Where is $f''(x)$ undefined? Nowhere

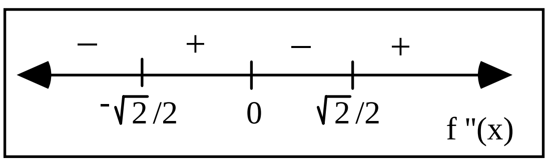
d) Sign analysis: Plot the numbers found above on a number line. Choose test values for each interval created and evaluate the second derivative.

$f''(-2) = 60(-2)^3 - 30(-2) = -420 < 0 \Rightarrow$ $f(x)$ **is concave down on** $\left(-\infty, -\frac{\sqrt{2}}{2}\right)$.

$f''(-\frac{1}{2}) = 60(-\frac{1}{2})^3 - 30(-\frac{1}{2}) = \frac{15}{2} > 0 \Rightarrow$ $f(x)$ **is concave up on** $\left(-\frac{\sqrt{2}}{2}, 0\right)$.

$f''(\frac{1}{2}) = 60(\frac{1}{2})^3 - 30(\frac{1}{2}) = -\frac{15}{2} < 0 \Rightarrow$ $f(x)$ **is concave down on** $\left(0, \frac{\sqrt{2}}{2}\right)$.

$f''(2) = 60(2)^3 - 30(2) = 420 > 0 \Rightarrow$ $f(x)$ **is concave up on** $\left(\frac{\sqrt{2}}{2}, \infty\right)$.



7. Find any **inflection points:**

a) For which values of x (found above) is $f(x)$ defined? $x = 0$, $x = \pm\frac{\sqrt{2}}{2}$

b) Find corresponding y-values:

$$f(0) = 3(0)^5 - 5(0)^3 = 0$$
$$f(\frac{\sqrt{2}}{2}) = 3(\frac{\sqrt{2}}{2})^5 - 5(\frac{\sqrt{2}}{2})^3 = -\frac{7\sqrt{2}}{8}$$
$$f(-\frac{\sqrt{2}}{2}) = 3(-\frac{\sqrt{2}}{2})^5 - 5(-\frac{\sqrt{2}}{2})^3 = \frac{7\sqrt{2}}{8}$$

c) $f(x)$ changes from concave down to concave up at $x = -\frac{\sqrt{2}}{2}$ so $\left(-\frac{\sqrt{2}}{2}, \frac{7\sqrt{2}}{8}\right)$ is an inflection point.

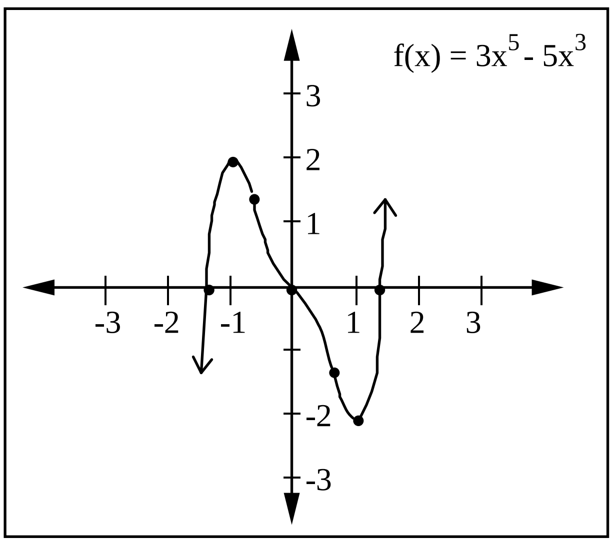
$f(x)$ changes from concave up to concave down at $x = 0$ so $(0,0)$ is an inflection point.

$f(x)$ changes from concave down to concave up at $x = \frac{\sqrt{2}}{2}$ so $\left(\frac{\sqrt{2}}{2}, -\frac{7\sqrt{2}}{8}\right)$ is an inflection point.

8. Note in a chart your points obtained:

x	y	
0	0	(y-intercept, inflection point)
-1	2	(maximum point)
1	-2	(minimum point)
$\sqrt{2}/2$	$-7\sqrt{2}/8$	(inflection point)
$-\sqrt{2}/2$	$7\sqrt{2}/8$	(inflection point)
$\sqrt{5/3}$	0	(x-intercept)
$-\sqrt{5/3}$	0	(x-intercept)

9. Plot all points on the coordinate plane, and sketch in the rest of the graph. Be sure to include all maximum points, minimum points, and inflection points:



Sample Problem #3:

$$f(x) = \frac{x^2+1}{x^2-9}$$

1. Look for any **asymptotes:**

a) Vertical: For which values of x is $f(x)$ undefined? (i.e. When is the denominator zero?)

$x^2 - 9 = 0$
$x^2 = 9$
$x = 3$ and $x = -3$

$\lim_{x\to -3^-} f(x) = \infty$ and $\lim_{x\to -3^+} f(x) = -\infty$ Therefore $x = -3$ is a vertical asymptote.

$\lim_{x\to 3^-} f(x) = -\infty$ and $\lim_{x\to 3^+} f(x) = \infty$ Therefore $x=3$ is a vertical asymptote.

b) Horizontal:

$$\lim_{x\to\pm\infty}\frac{x^2+1}{x^2-9} = \lim_{x\to\pm\infty}\frac{x^2+1}{x^2-9}\cdot\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right) = \lim_{x\to\pm\infty}\frac{\frac{x^2}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}-\frac{9}{x^2}} = \lim_{x\to\pm\infty}\frac{1+\frac{1}{x^2}}{1-\frac{9}{x^2}} = \lim_{x\to\pm\infty}\frac{1+0}{1-0} = \lim_{x\to\pm\infty}\frac{1}{1} = 1$$

Therefore $y=1$ is a horizontal asymptote.

2. **Intercepts:**

a) y-intercepts: $f(0) = -\frac{1}{9}$ **y-intercept:** $\left(0, -\frac{1}{9}\right)$

b) x-intercepts: $0 = \frac{x^2+1}{x^2-9}$ only where $0 = x^2+1$

There is no x-intercept (because the numerator is always positive).

3. **Increasing/decreasing:**

a) Take the first derivative: $f'(x) = \frac{(2x)(x^2-9)-(x^2+1)(2x)}{(x^2-9)^2}$

$= \frac{2x^3-18x-2x^3-2x}{(x^2-9)^2}$

b) Set it equal to zero: $= \frac{-20x}{(x^2-9)^2} = 0$

c) Solve for x. (When does the numerator = 0?) $-20x=0$ so $x=0$

d) Where is $f'(x)$ undefined? (When does the denominator =0?) $x=3$ and $x=-3$

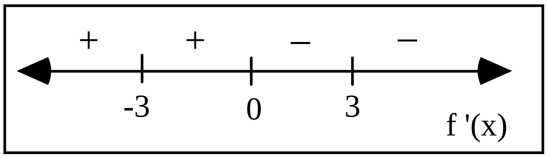
e) Sign analysis:

$f'(-5) = \frac{-20(-5)}{((-5)^2-9)^2} = \frac{100}{256} = \frac{25}{64}$ Positive $\Rightarrow$ $f(x)$ **is increasing on** $(-\infty,-3)$.

$f'(-1) = \frac{-20(-1)}{((-1)^2-9)^2} = \frac{20}{64} = \frac{5}{16}$ Positive $\Rightarrow$ $f(x)$ **is increasing on** $(-3,0)$.

$f'(1) = \frac{-20(1)}{((1)^2-9)^2} = -\frac{20}{64} = -\frac{5}{16}$ Negative $\Rightarrow$ $f(x)$ **is decreasing on** $(0,3)$.

$f'(5) = \frac{-20(5)}{((5)^2-9)^2} = -\frac{100}{256} = -\frac{25}{64}$ Negative $\Rightarrow$ $f(x)$ **is decreasing on** $(3,\infty)$.



4. **Extrema:**

a) For which values of x (found above in 3) is $f(x)$ defined? $x = 0$

b) Find corresponding values of y: $f(0) = -\frac{1}{9}$

c) Critical point: $\left(0, -\frac{1}{9}\right)$

d) Extrema: $f'(x)$ changes from positive to negative at $x = 0$ so $\left(0, -\frac{1}{9}\right)$ is a local extremum of f.

5. Test extrema for local **max/mins:**

<u>SECOND DERIVATIVE TEST</u>

a) Take the second derivative:

$$f''(x) = \frac{-20(x^2-9)^2 - (-20x)(2(x^2-9)(2x))}{(x^2-9)^4}$$

$$f''(x) = \frac{60x^2+180}{(x^2-9)^3}$$

b) Substitute x-coordinate of extremum: $f''(0) = \frac{60(0)^2+180}{((0)^2-9)^3} = -\frac{20}{81}$ (negative $\Rightarrow$ max)

c) Label your points: $\left(0, -\frac{1}{9}\right)$ **local maximum**

or:

<u>FIRST DERIVATIVE TEST</u>

a) $f(x)$ is increasing before $x = 0$ and decreasing after $x = 0$, so
$\left(0, -\frac{1}{9}\right)$ **local maximum**

6. **Concave up/concave down:**

a) Set $f''(x)$ equal to 0: $\frac{60x^2+180}{(x^2-9)^3} = 0$

b) Solve for x. (When does the numerator = 0?) $60\left(x^2+3\right) = 0$
x^2+3 is never 0, so no solutions here.

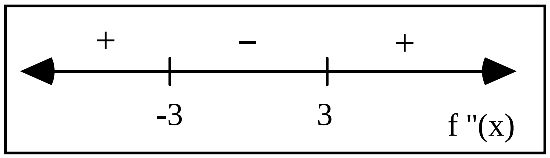
c) Where is $f''(x)$ undefined? $x = 3$ and $x = -3$

d) Sign analysis:

$$f''(-4) = \frac{60(-4)^2 + 180}{((-4)^2 - 9)^3} = \frac{1{,}140}{343}$$ positive $\Rightarrow$ $f(x)$ **concave up on** $(-\infty, -3)$.

$$f''(0) = \frac{60(0)^2 + 180}{((0)^2 - 9)^3} = -\frac{20}{81}$$ negative $\Rightarrow$ $f(x)$ **concave down on** $(-3, 3)$.

$$f''(4) = \frac{60(4)^2 + 180}{((4)^2 - 9)^3} = \frac{1{,}140}{343}$$ positive $\Rightarrow$ $f(x)$ **concave up on** $(3, \infty)$.



7. Find any **inflection points:**

a) for which values of x (found in 6) is $f(x)$ defined? $f(x)$ is undefined at $x = -3$ and at $x = 3$ thus we can have no inflection points.

8. Note in a chart your points obtained

x	y	
0	-1/9	(maximum point.)

9. Plot all points and asymptotes on the coordinate plane and sketch in the rest of the graph using the information found above.